Dynamics of singularities and wavebreaking in 2D hydrodynamics with free surface

Pavel Lushnikov

Department of Mathematics and Statistics, University of New Mexico, USA

Support: NSF DMS-0807131, NSF PHY-1004118, NSF DMS-141214

UNM

Collaborators:

Sergey Dyachenko[1], Alexander Korotkevich[2], and Denis A. Silantyev[2]

[1]**Brown University, USA**
[2]**Department of Mathematics and Statistics, University of New Mexico, USA**

3D Euler's equations of incompressible fluid motion in gravitational field g

$$\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{1}{\rho}\nabla p + \mathbf{g} = 0$$

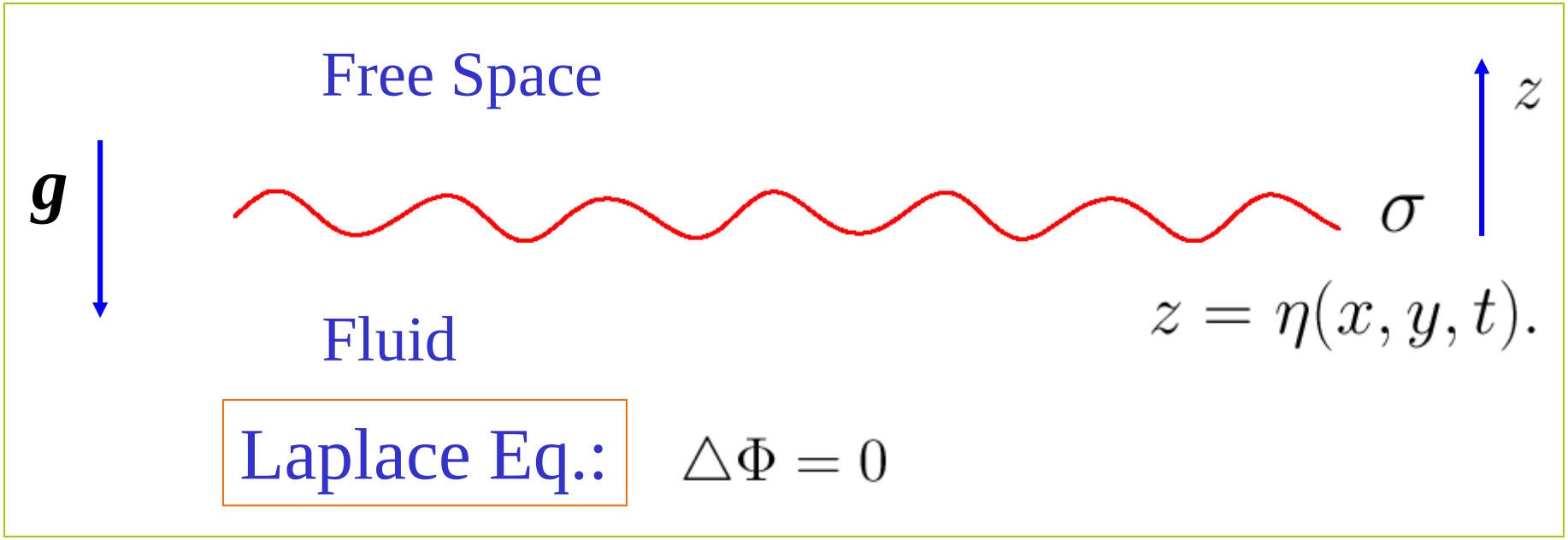
$$\nabla\cdot\mathbf{v} = 0$$

Reduction: potential flow

$$\mathbf{v} = \nabla\Phi \qquad \Longrightarrow \qquad \nabla\cdot\mathbf{v} = \Delta\Phi = 0$$ - Laplace Eq.

$$\nabla\left[\Phi_t + \frac{(\nabla\Phi)^2}{2} + \frac{1}{\rho}p + gz\right] = 0$$ - Bernouilli Eq.

Free surface hydrodynamics



$\boldsymbol{g}$ - acceleration of gravity

σ - surface tension coefficient

$z = \eta(x, y, t)$ - shape of free surface

$\Phi_z|_{z=-h} = 0$ - boundary condition at the bottom

Boundary conditions at free surface:

Kinematic condition:

$$\frac{d\eta}{dt} = \frac{\partial \eta}{\partial t} + (\mathbf{v} \cdot \nabla)\eta = v_z$$

vertical component of velocity

Dynamic boundary condition:

$$p\Big|_{z=\eta} = \sigma \nabla \cdot \frac{\nabla \eta}{\sqrt{1 + (\nabla \eta)^2}}$$

$p\Big|_{z=\eta}$ - pressure at free surface $z = \eta(x, y, t)$.

Bernouilli Eq.:

$$\Phi_t + \frac{1}{2}\left(\nabla \Phi\right)^2 + p + gz = 0$$

Kinematic and dynamic boundary conditions together with Laplace Eqs. $\boxed{\triangle\Phi = 0}$ form a closed set of equations.

Equivalent Hamiltonian formulation (Zakharov, 1968):

$$\frac{\partial\Psi}{\partial t} = -\frac{\delta H}{\delta\eta},$$
$$\frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\Psi},$$

where $\Psi \equiv \Phi\Big|_{z=\eta}$ - velocity potential at free surface

The Hamiltonian =kinetic energy+ potential energy, $H = T + U$

$$T = \frac{1}{2}\int d\mathbf{r}\int_{-h}^{\eta}\left(\nabla\Phi\right)^2 dz,$$

$$U = \frac{1}{2}g\int \eta^2\, d\mathbf{r} + \sigma\int\left[\sqrt{1+\left(\nabla\eta\right)^2}-1\right]d\mathbf{r}$$

potential energy in the gravitational field

surface tension energy

The Hamiltonian can be rewritten as a surface integral:

$$H = \frac{1}{2}\int \Big[V_n \Psi \sqrt{1+(\nabla\eta)^2} + g\eta^2 + 2\sigma\big(\sqrt{1+(\nabla\eta)^2}-1\big)\Big]\, d^2\mathbf{r}$$

$$\mathbf{r} = (x, y),\ \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right).$$

Normal velocity component: $V_n = \mathbf{n}\cdot\nabla\Phi$

Unit normal vector: $\mathbf{n} = (-\nabla\eta, 1)\dfrac{1}{\sqrt{1+(\nabla\eta)^2}}$

The Hamiltonian perturbation theory:

The Hamiltonian H depend on the normal velocity V_n which has to be expressed in terms of canonical variables Ψ and η .
But $\Psi := \Phi|_{z=\eta}$ is the Dirichlet boundary condition for Φ

while V_n s the Neumann boundary condition, $V_n := \mathbf{n} \cdot \nabla\Phi|_{z=\eta}$, for Φ .

It means that we have to solve the Laplace Eq. $\triangle\Phi = 0$ With the Dirichlet $\Psi := \Phi|_{z=\eta}$ boundary condition to find V_n .

In other words, it is necessary to determine Dirichlet-Neumann operator $\hat{G}\Psi = \left[1 + (\nabla\eta)^2\right]^{1/2} \mathbf{n} \cdot \nabla\Phi|_{z=\eta}$

which relates V_n and Ψ .

Perturbation technique:

Flat free surface is stable.

Series expansion of V_n in powers of Ψ and η allows to develop a perturbation theory for small deviations from flat surface.

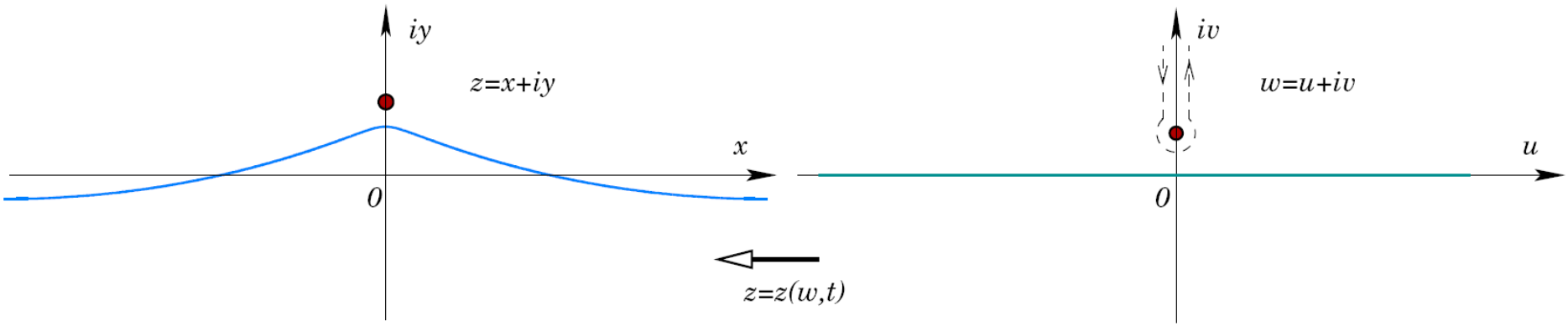
Small parameter of perturbation theory: $|\nabla\eta|$ - a typical slope of surface elevation.

For strongly nonlinear solutions one cannot use the perturbation theory. Instead we use the complex form of 2D hydrodynamics with free surface to explicitly solve the Laplace Eq. $\triangle\Phi = 0$ at each moment of time.

Free surface parametrization in 2D: $y = \eta(x,t)$

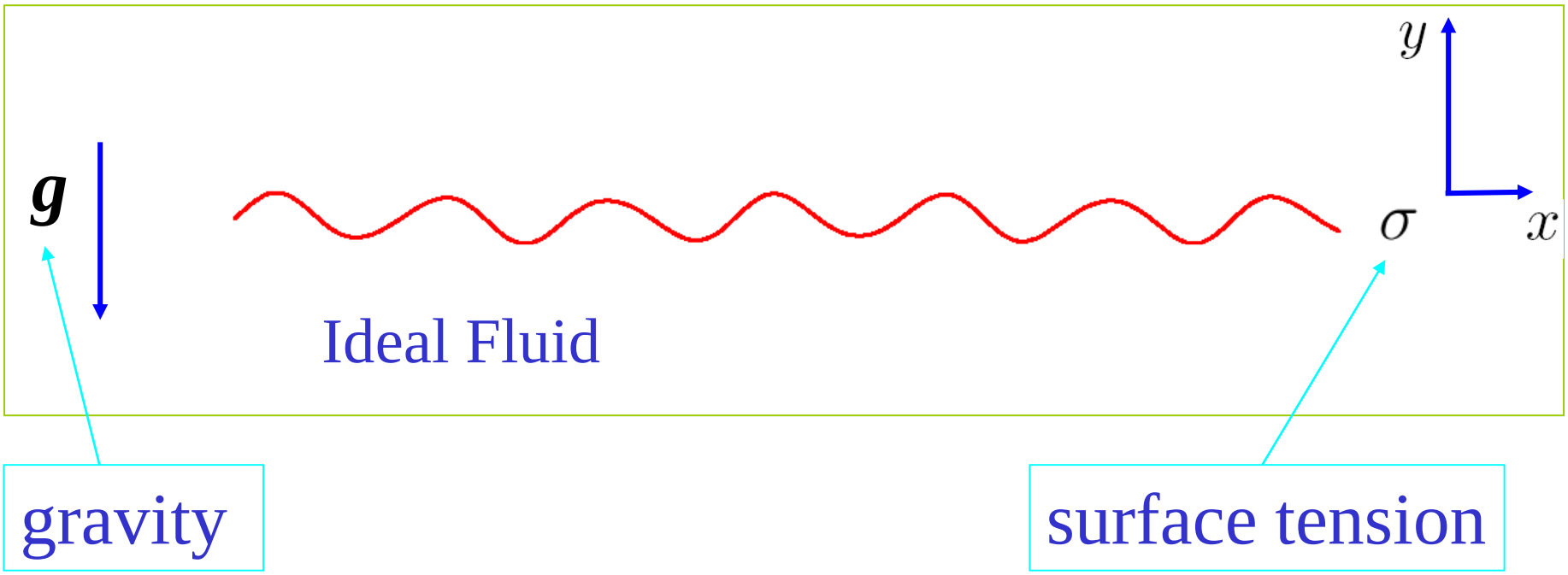
Complex variable: $z = x + iy$

Conformal map from lower complex half-plane of $w = u + iv$ into fluid domain $z = x + iy$



$$y = \eta(x,t) \Longleftarrow v = 0$$

2D Hydrodynamics of ideal fluid with free surface



$y = \eta(x, t)$ - shape of free surface

Stream function Θ is defined by

$$\frac{\partial}{\partial x}\Theta = -\frac{\partial}{\partial y}\Phi = -v_y \text{ and } \frac{\partial}{\partial y}\Theta = \frac{\partial}{\partial x}\Phi = v_x$$

which ensures the incompressibility condition:

$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x}v_x + \frac{\partial}{\partial y}v_y = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\Theta + \frac{\partial}{\partial y}\left[-\frac{\partial}{\partial x}\Theta\right] = 0$$

Define complex potential as $\Pi = \Phi + \mathrm{i}\Theta$

then $\frac{\partial}{\partial x}\Theta = -\frac{\partial}{\partial y}\Phi = -v_y$ and $\frac{\partial}{\partial y}\Theta = \frac{\partial}{\partial x}\Phi = v_x$
turns into Cauchy-Riemann conditions for analyticity of

$$\Pi(z),\ z = x + \mathrm{i}y$$

The complex velocity: $V := \Pi'(z),\ V = v_x - \mathrm{i}v_y$

Fluid dynamics in conformal variables (exact form of Euler equation for fluid with free surface)[1]:

$$y_t = (y_u \hat{H} - x_u)\frac{\hat{H}\Psi_u}{|z_u|^2} \qquad z = x + iy$$

$$\Psi_t = \frac{\hat{H}(\Psi_u \hat{H}\Psi_u)}{|z_u|^2} + \Psi_u \hat{H}\left(\frac{\hat{H}\Psi_u}{|z_u|^2}\right) - gy + \sigma \frac{1}{x_u}\frac{\partial}{\partial u}\frac{y_u}{|z_u|}$$

$$x = u - \hat{H}y$$

Hilbert transform:

$$\hat{H}f(u) = \frac{1}{\pi}\text{p.v.}\int_{-\infty}^{+\infty}\frac{f(u')}{u'-u}\mathrm{d}u'$$

Hilbert transform in Fourier domain: $i\,\mathrm{sign}(k)$

---

[1]A.I. Dyachenko, E.A. Kuznetsov, M. Spector and V.E. Zakharov, Phys. Lett. A **221**, 73 (1996).

Water waves even in **2D** are not integrable (fourth order matrix element is zero while 5th order is not zero on resonance surfaces) [1].

Instead we suggest to fully describe **2D** hydrodynamics of idea fluid with free surface by the dynamics of complex singularities outside of fluid.

---

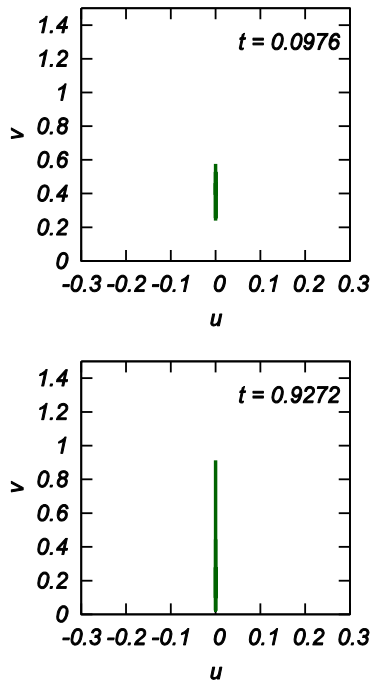
[1] A.I. Dyachenko, Y.V. Lvov and V.E. Zakharov, Phys. D **87**, 233-261 (1995).

Example: Motion of branch cut for zero gravity

Weakly nonlinear solution[1]

$$V(w) = \frac{d\Pi(w)}{dw} = \frac{2A}{w - \mathrm{i}a + \sqrt{(w - \mathrm{i}a)^2 - 8At}}$$

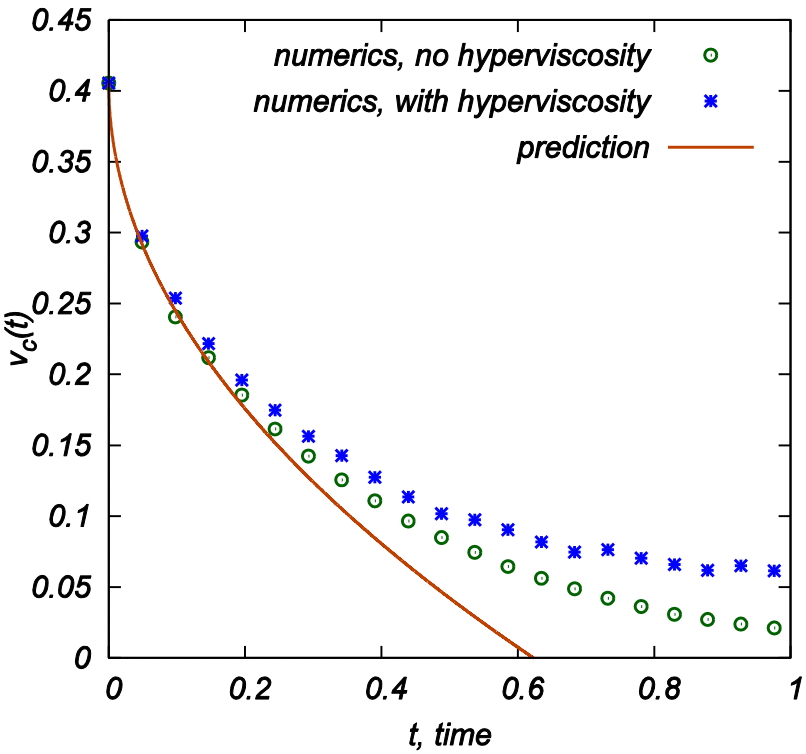
Complex velocity potential



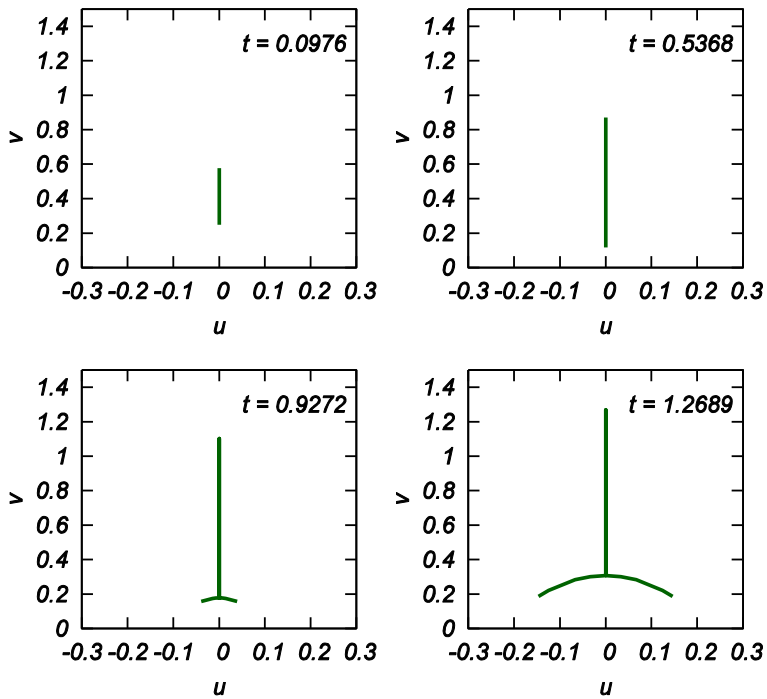
Brunch cut approaches and later hits free surface

[1]E. A. Kuznetsov, M. D. Spector, and V. E. Zakharov. Phys. Rev. E, 49:1283–1290 (1994).

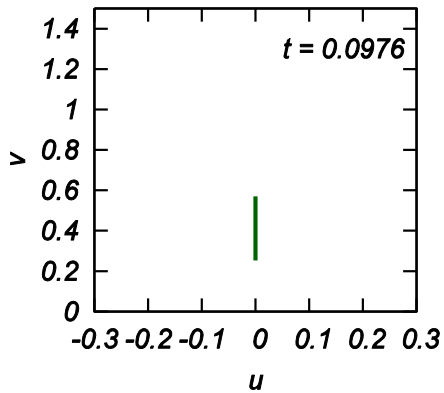
Distance from lower end of branch cut vs. time for weakly nonlinear (red line) and fully nonlinear soltion (green circles)

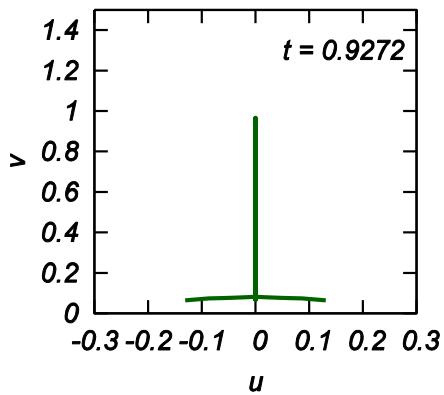


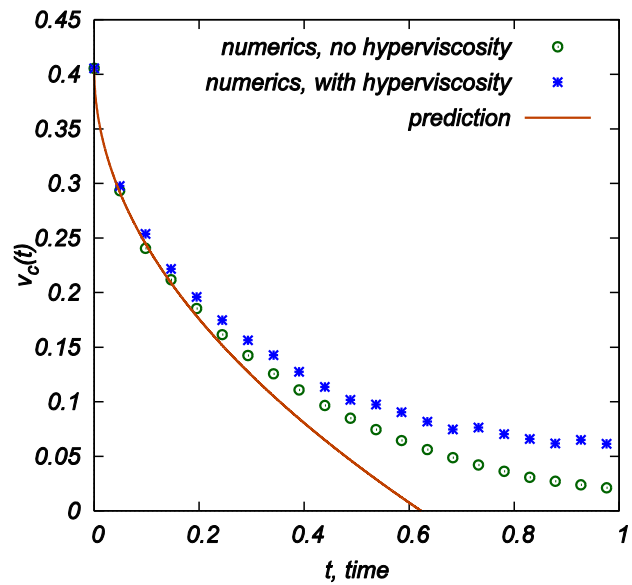
Addition of gravity causes bifurcation of the initially vertical branch cut into side branches



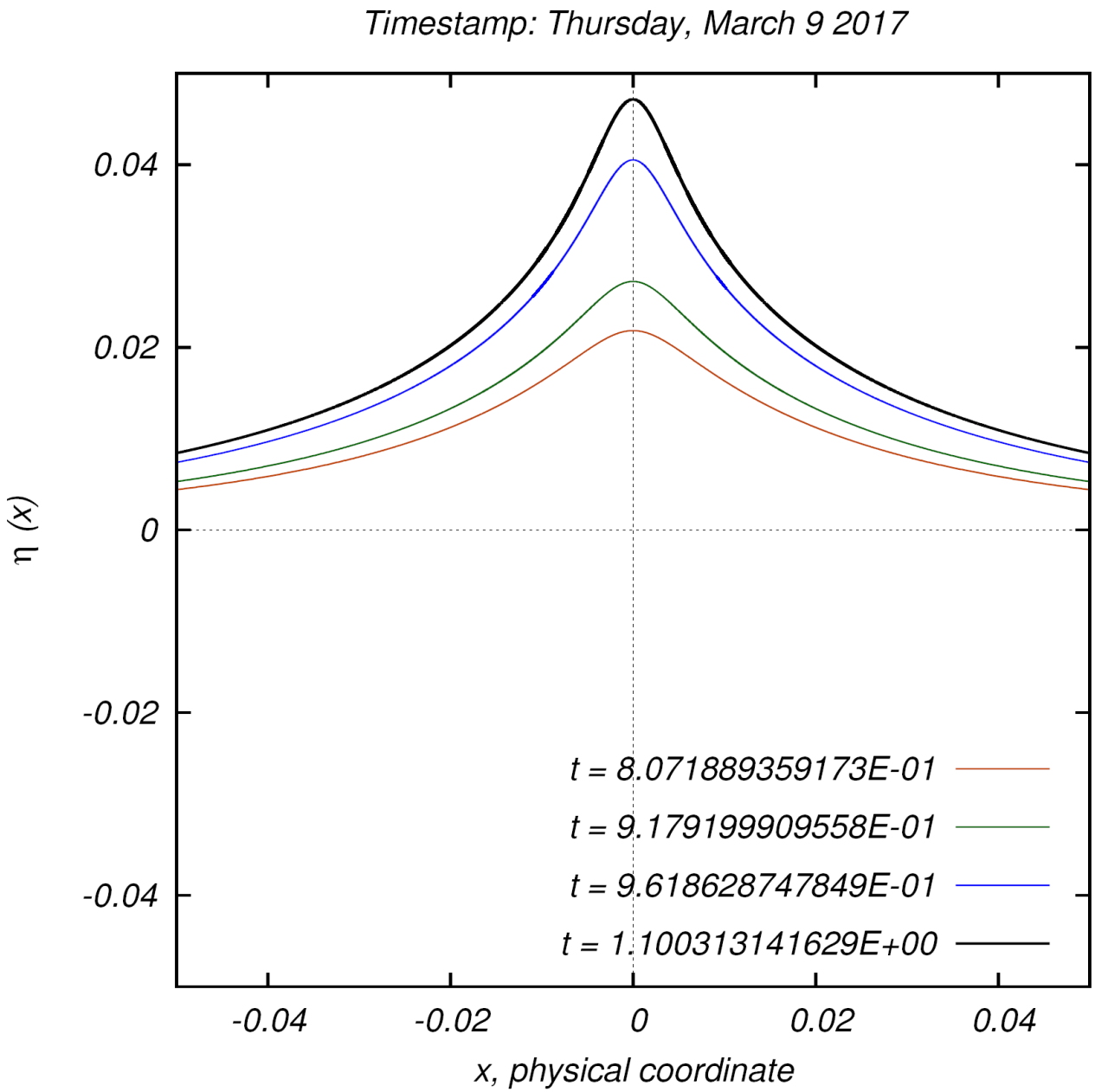
The addition of hypervisosity (instead of gravity) is expected to
Regularizes wavebreaking but causes the forking of the initially
Vertical branch cut qualitatively similar to gravity case







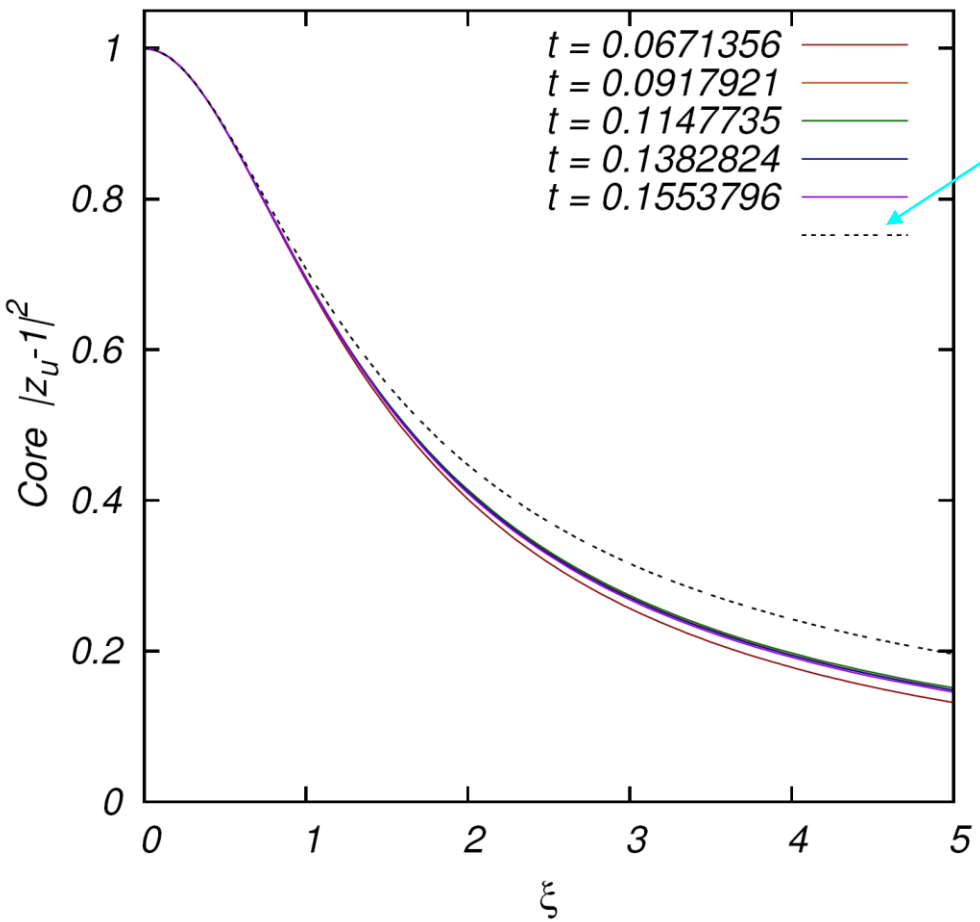
Spatial profile in physical coordinates



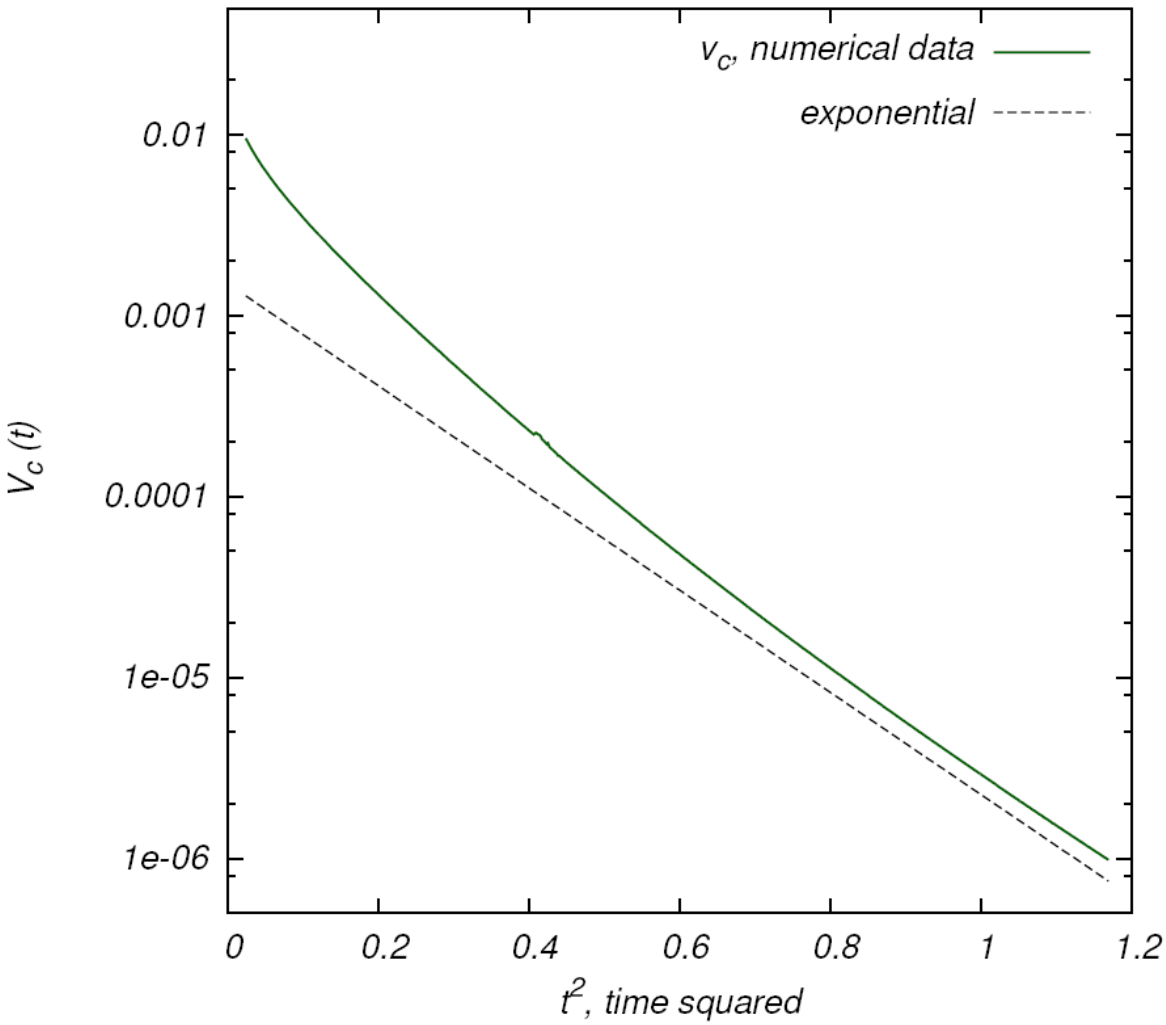
Rescaling to self-similar solution

$$|z_u|^2 = z_u \bar{z}_u = \frac{c^2}{4} \frac{1}{\sqrt{w^2 + v_c^2}}$$

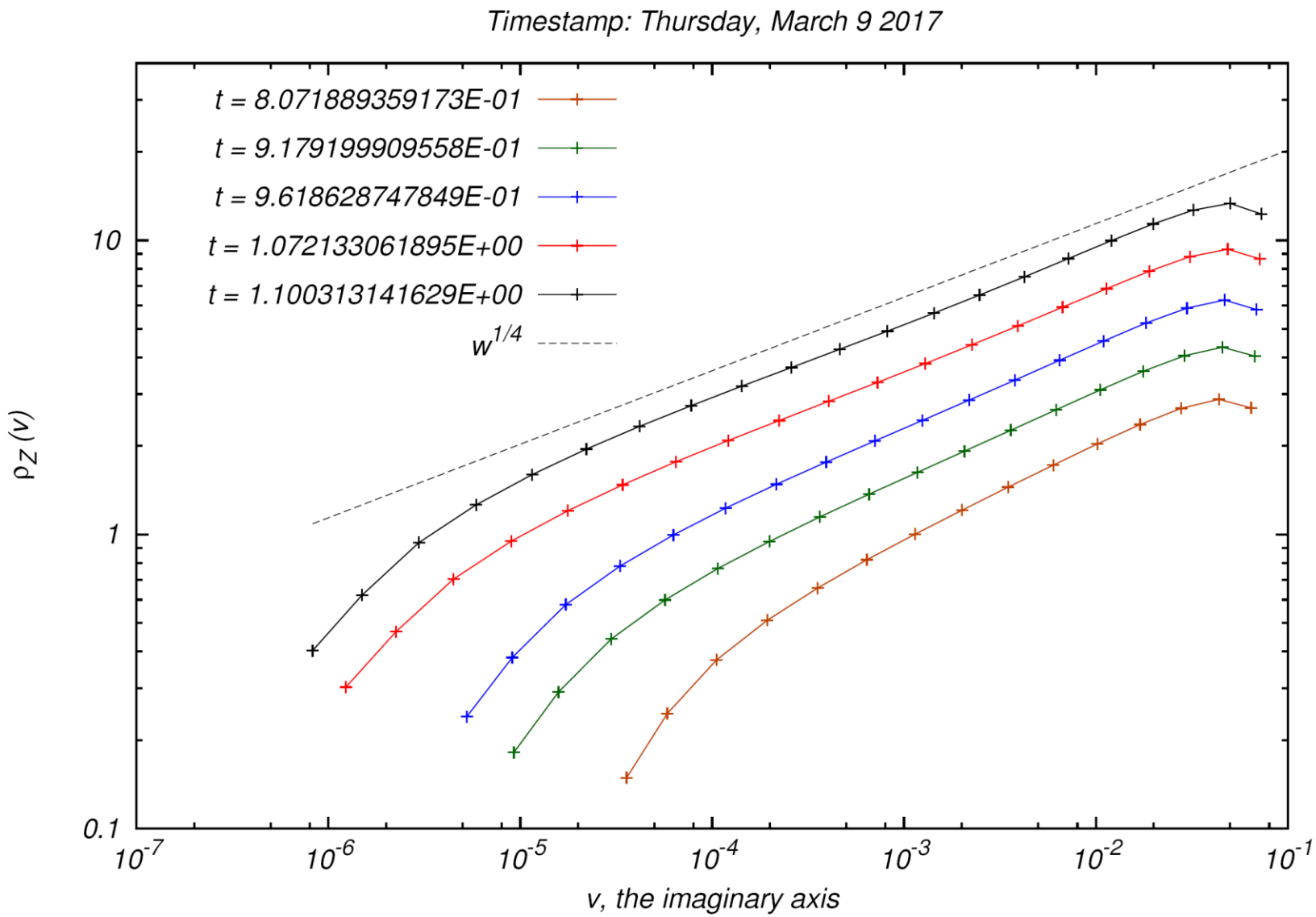
$$\xi = \frac{w}{v_c(t)}$$



Time dependence $v_c(t) \propto e^{-\alpha t^2}$



Jump at the branch cut



Particular case: Travelling wave (Stokes wave) with zero capillarity $\sigma = 0$

Travelling wave implies the solution in the following form

$$\tilde{z} = \tilde{z}(u - ct)$$

$$\psi = \psi(u - ct)$$

Here $x = u + \tilde{x}(u, t)$

$$\tilde{z} = \tilde{x} + iy$$

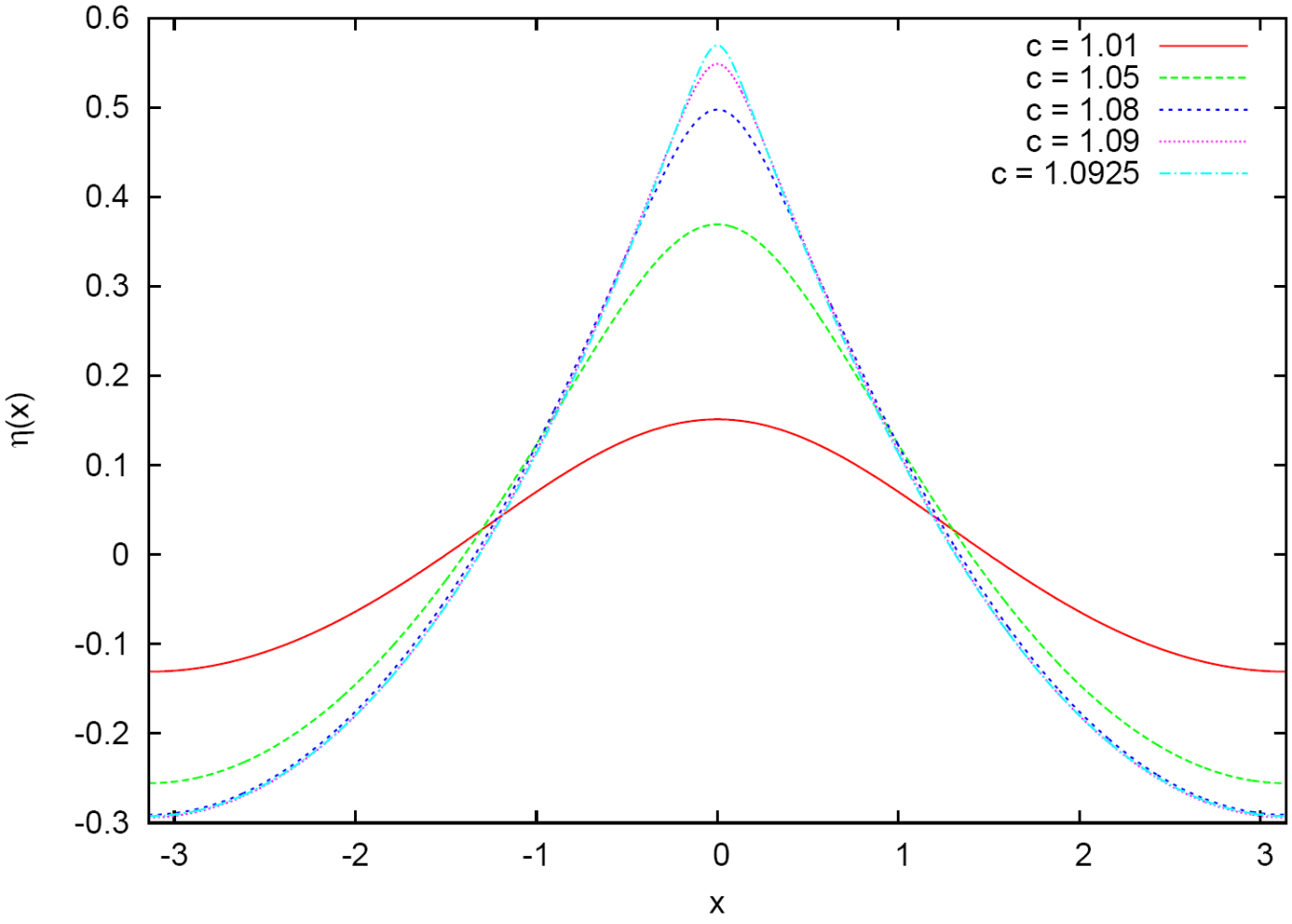
$\Rightarrow$ Dynamical equations are reduced to

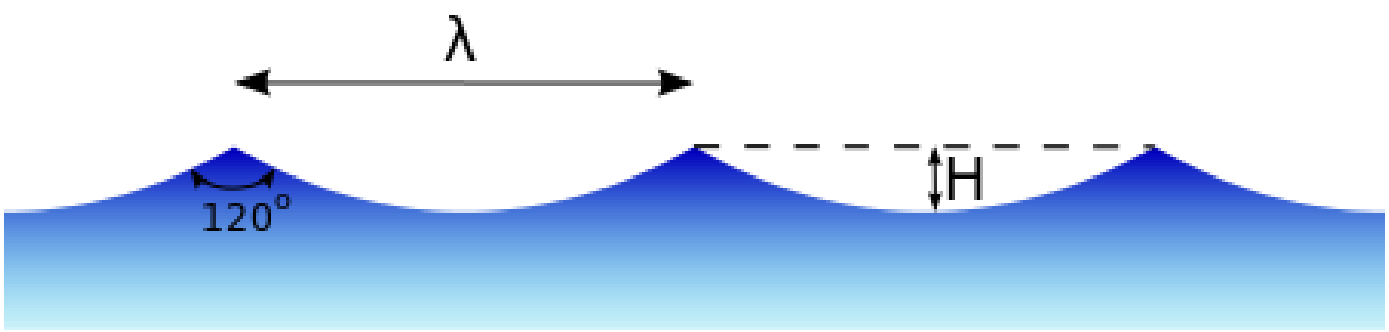
$$\boxed{(c^2\hat{k} - 1)y - \left(\frac{\hat{k}y^2}{2} + y\hat{k}y\right) = 0}$$

$$\hat{k} := -\frac{\partial}{\partial u}\hat{H}$$

Hilbert transform

Stokes wave for different velocities *c* with *g=1*





H / λ ≈ 0.1410633…

Low amplitude limit of Stokes wave[1]

$$\eta(x,t) = a\left\{\cos\theta + \tfrac{1}{2}(ka)\,\cos 2\theta + \tfrac{3}{8}(ka)^2\,\cos 3\theta\right\} + O\left((ka)^4\right),$$

$$\phi(x,y,t) = a\frac{\omega}{k}\,\mathrm{e}^{ky}\,\sin\theta + O\left((ka)^4\right),$$

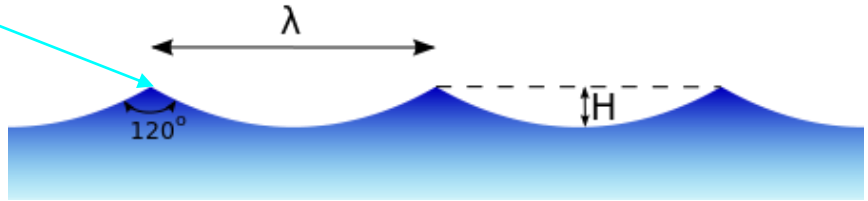
$$c = \frac{\omega}{k} = \left(1 + \tfrac{1}{2}(ka)^2\right)\sqrt{\frac{g}{k}} + O\left((ka)^4\right)$$

$$\theta(x,t) = kx - \omega t,$$

[1]G. G. Stokes, Trans. Cambridge Philos. Soc. 8, 441 (1847).

Limiting Stokes wave (wave of maximum height)[1]

$$z = c_0 + e^{-i\pi/6}\frac{3^{2/3}}{2^{2/3}}w^{2/3}$$



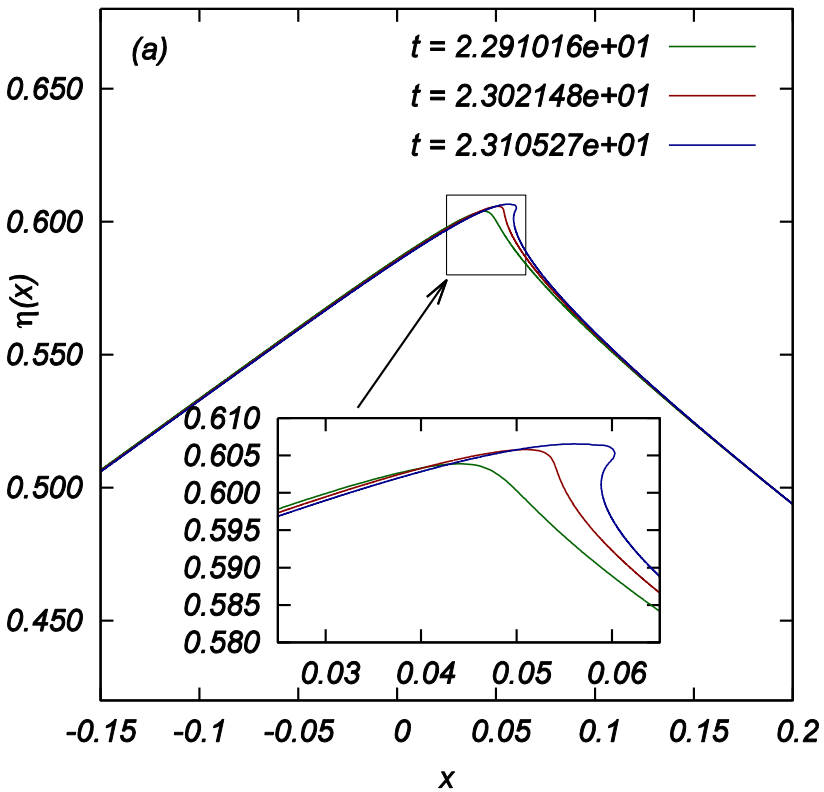
Next order correction[2]

$$z = c_0 + e^{-i\pi/6}\frac{3^{2/3}}{2^{2/3}}w^{2/3} + c_1 w^{\mu}\ldots, \quad \mu = 1.469345740433356\ldots$$

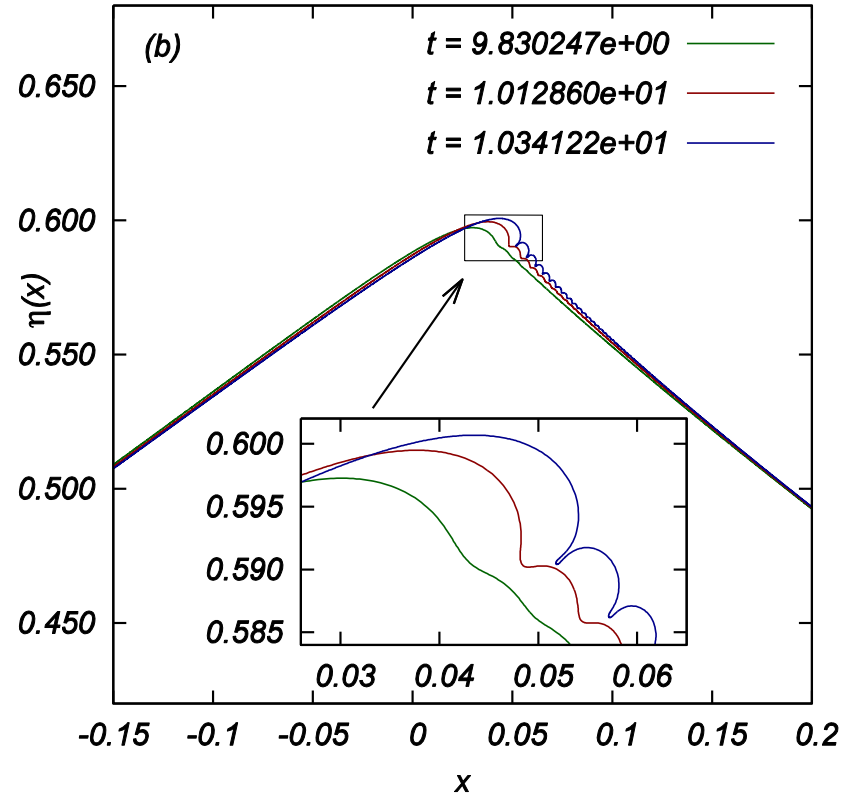
---

[1]G. G. Stokes, Math. Phys. Pap. **1**, 197 (1880).
[2]M. A. Grant, J. Fluid Mech. **59**, 257 (1973).

Adding capillarity or perturbing Stokes wave results in wavebreaking[1,2]



$\sigma = 0$

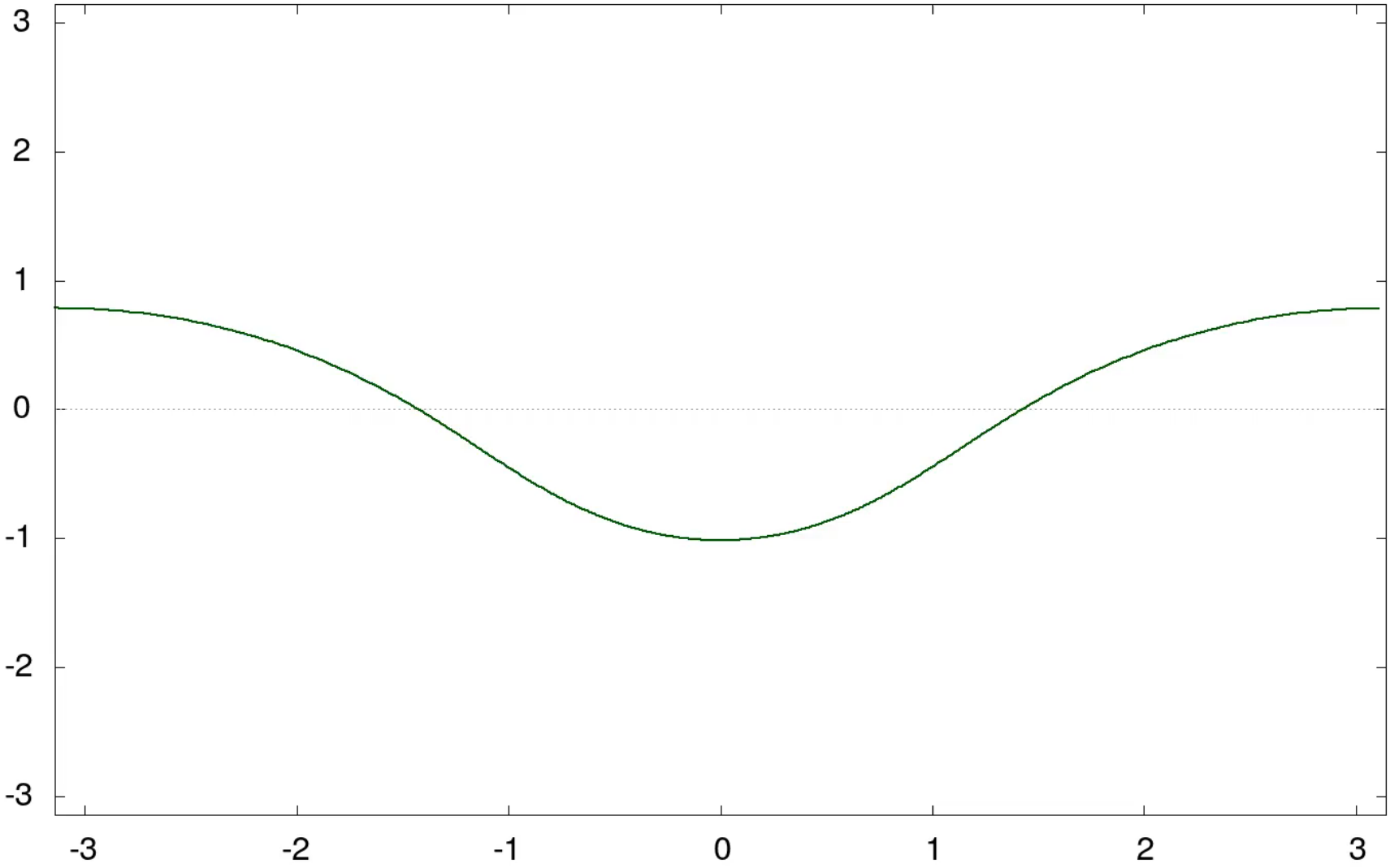


$\sigma = 1.6 \cdot 10^{-5}$

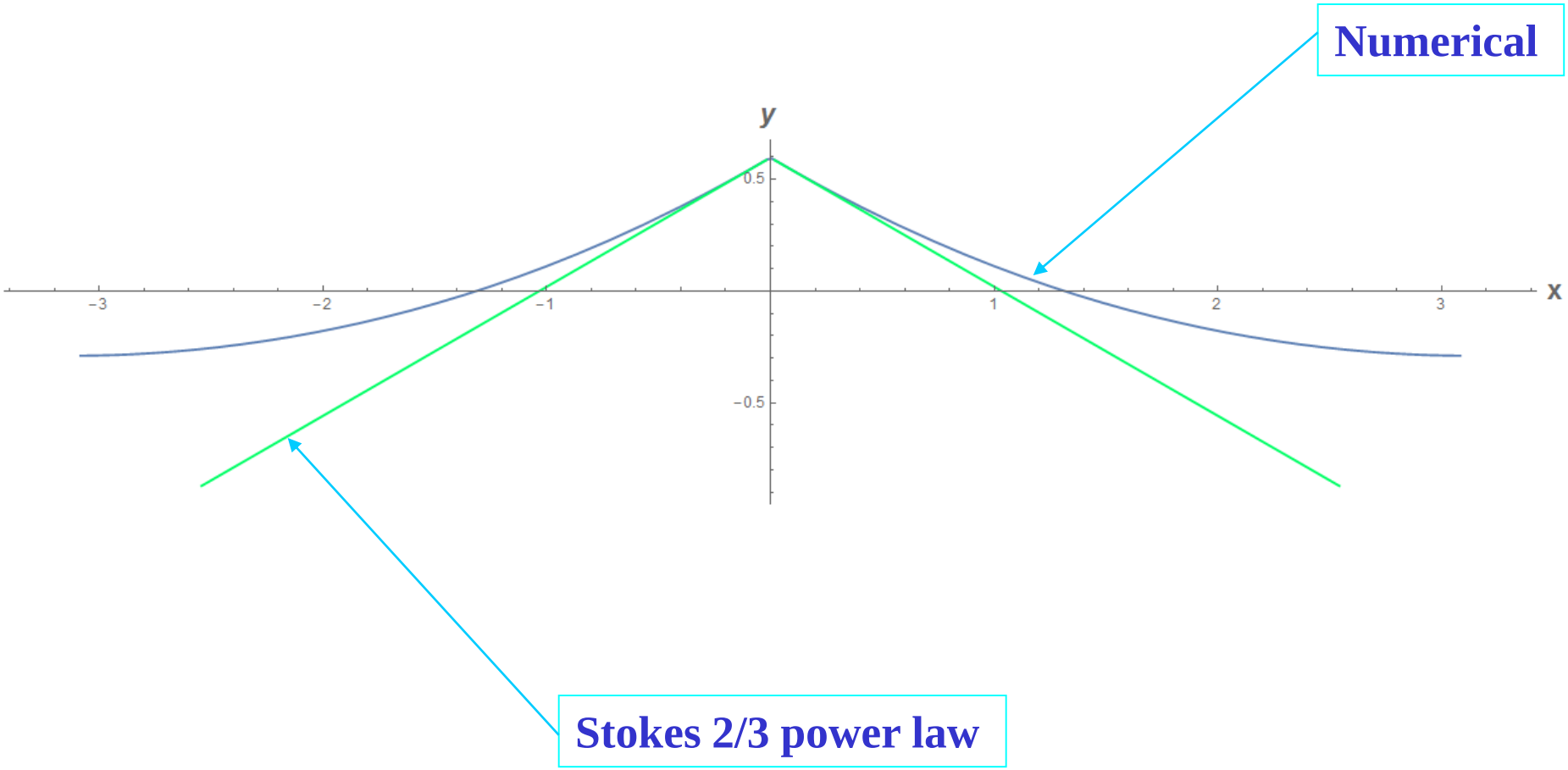
---

[1]S. A. Dyachenko and A.C. Newell, Stud. Appl. Math (2016)
[2]S. A. Dyachenko and P.M. Lushnikov (2016)

Plunging of overturning wave



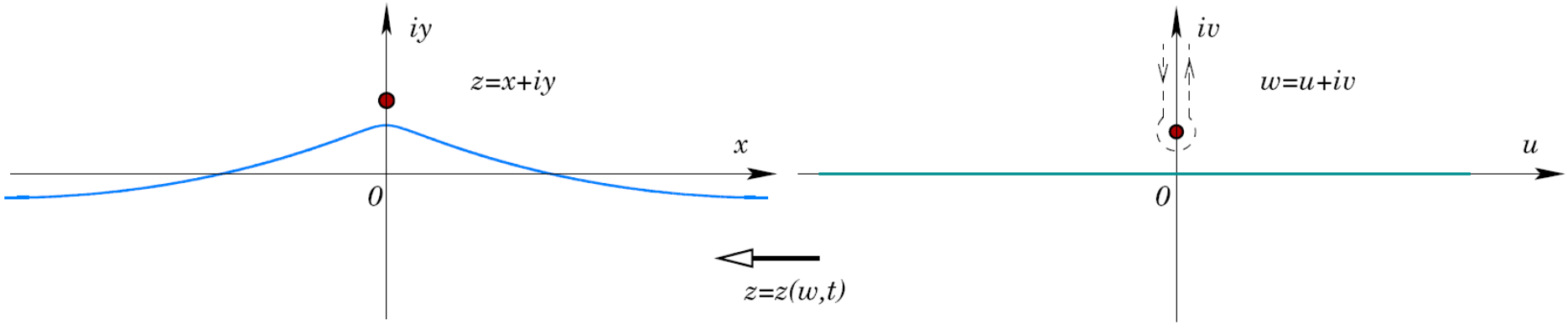
How non-limiting Stokes wave approach its limiting form?



$$z = c_0 + e^{-i\pi/6}\frac{3^{2/3}}{2^{2/3}}w^{2/3}$$

We look at Stokes wave through its complex singularities and how they approach real line [1]

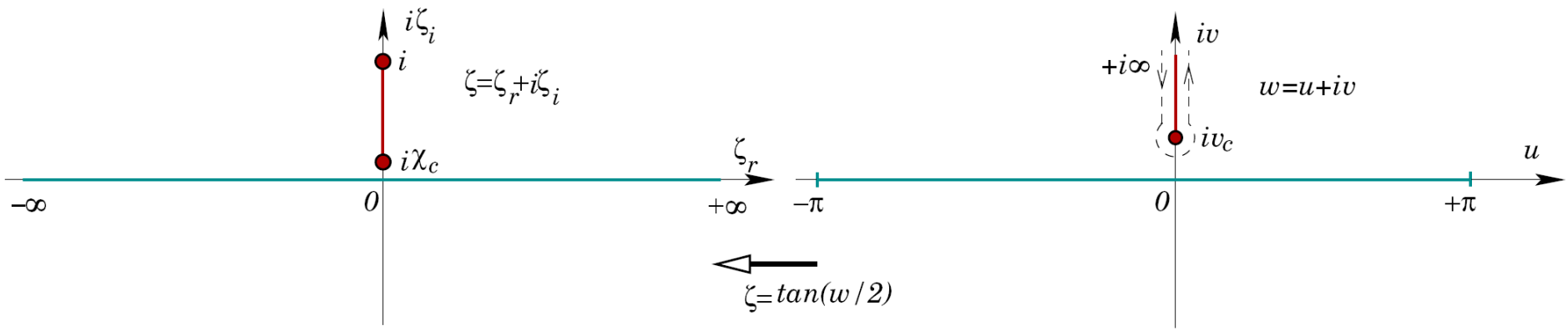
First conformal transform



[1] S.A. Dyachenko, P.M. Lushnikov, and A.O. Korotkevich, JETP Letters, v. 98, 675-679 (2014).

Second conformal transform to take into account spatial periodicity

$$\zeta = \kappa + i\chi = \tan\left(\frac{w}{2}\right)$$



$$\chi_c = \tanh\frac{v_c}{2}$$

Maps $u \in [-\pi, \pi]$ to the real line $\kappa \in (-\infty, \infty), \chi = 0$

$$i\infty \to i, \qquad -i\infty \to -i$$

Complex form of equation for Stokes wave

$$\boxed{(c^2\hat{k}-1)y-\left(\frac{\hat{k}y^2}{2}+y\hat{k}y\right)=0}$$

$$\Rightarrow \qquad \boxed{z_u = 1+\frac{1}{c^2}\hat{P}\left[(z-\bar{z})z_u\right]}$$

$\hat{P}=\frac{1+i\hat{H}}{2}$ - Projector to a function analytic in lower half plane

Two equivalent forms of equation for Stokes wave

(1) $$\boxed{z_u = 1 + \frac{1}{c^2}\hat{P}\left[(z - \bar{z})z_u\right]}$$

(2) $$y = -\frac{i}{2}(\tilde{z} - \bar{\tilde{z}}) = \frac{c^2}{2}\left(1 - \frac{1}{|z_u|^2}\right)$$

$\Rightarrow$ $$z_u = \frac{c^2}{\bar{z}_u[\mathrm{i}(z - \bar{z}) + c^2]}$$ - nonlinear ODE for z if $\bar{z}$ is known

But: non-Limiting Stokes wave can have only square root singularities[1]

$$\tilde{z} = \sum_{j=0}^{\infty} a_j \left(\zeta - i\chi_c\right)^{j/2}$$

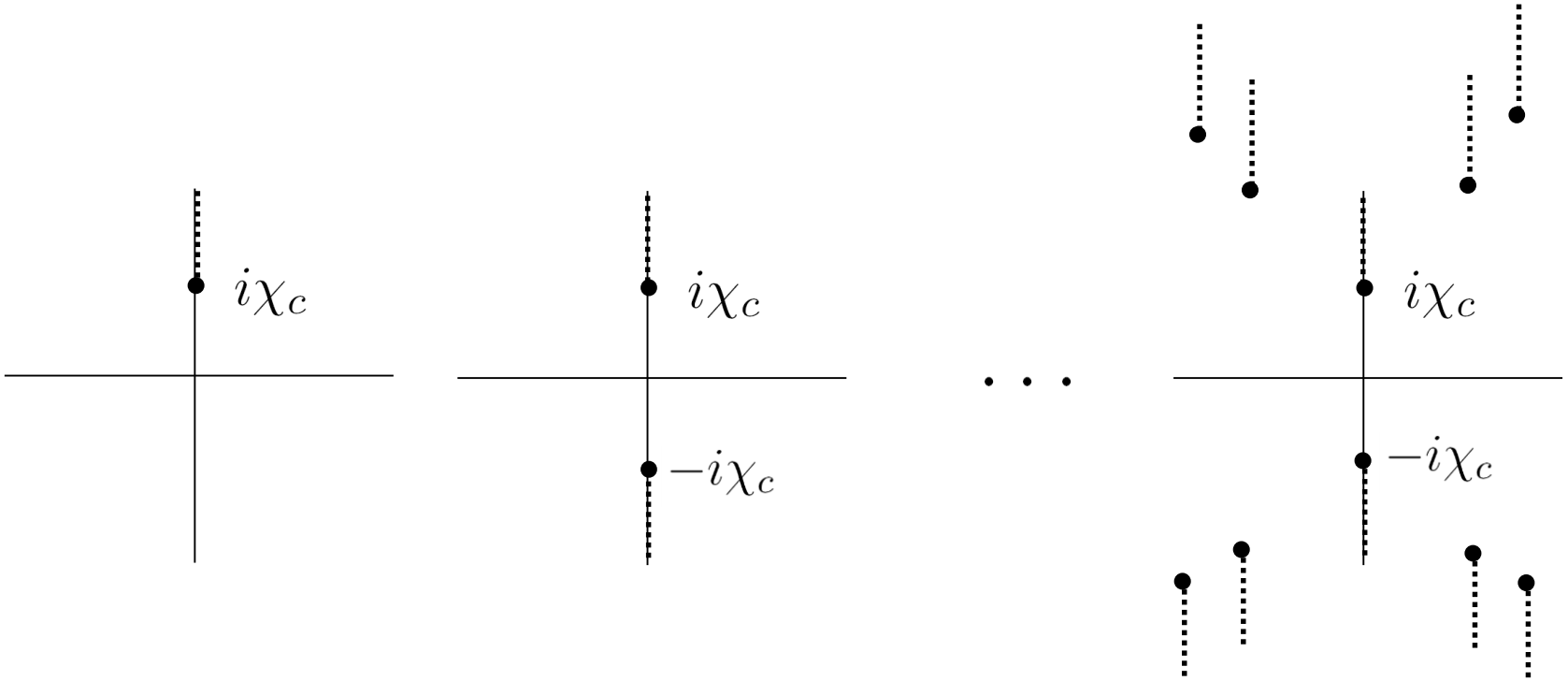
[1] S. Tanveer, Proc. R. Soc. Lond. A 435, 137-158 (1991).

Location of singularities in infinite numbers of sheets of Riemann surface[1]

First (physical) sheet

Second (non-physical) sheet

Third and higher sheets

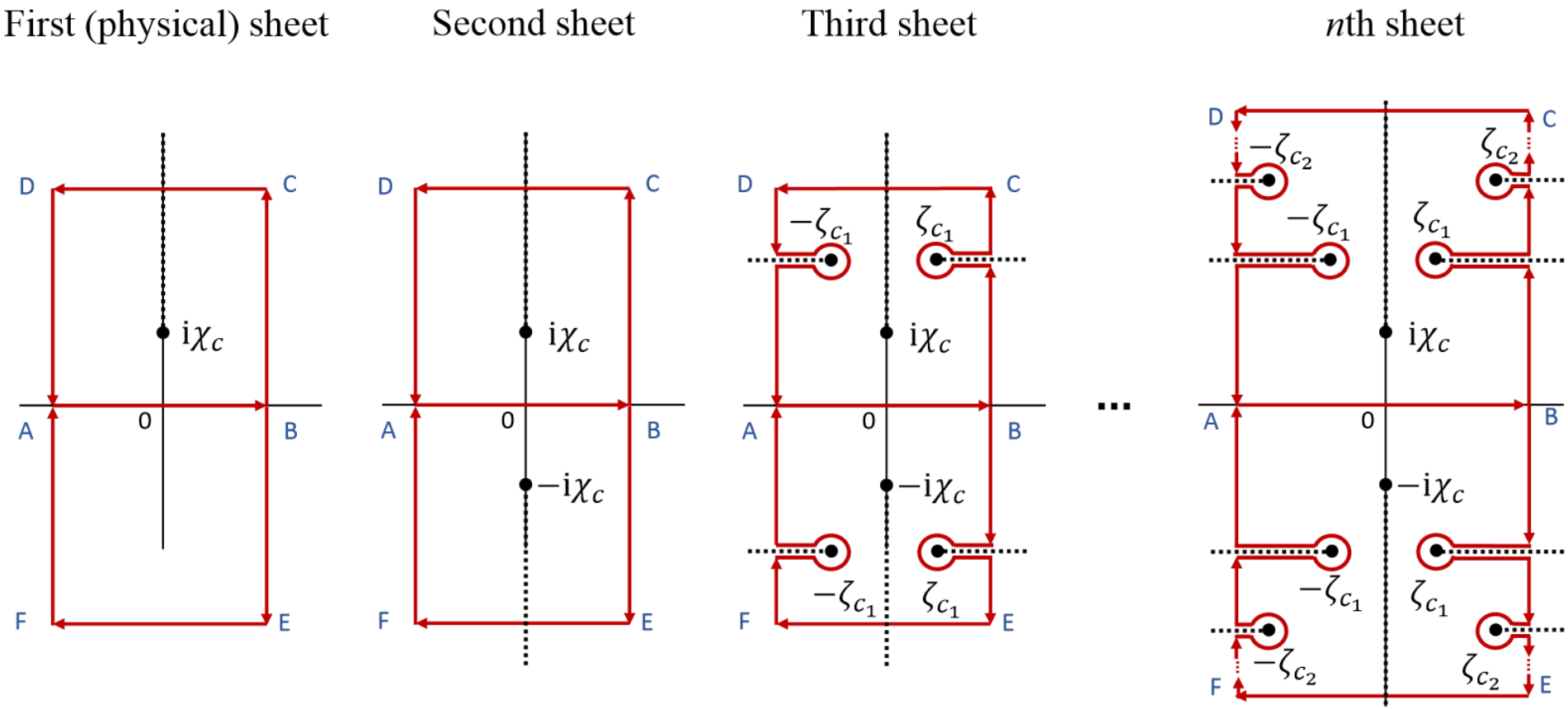


All singularities are square roots[1]

[1]P. M. Lushnikov, Journal of Fluid Mechanics, **800**, 557-594 (2016)

Two complementary approaches to analyze multiple sheets of Riemann Surface [1] :

Approach 1: Use ODE integration along complex contours for the second form of Stokes wave equation:



[1]P. M. Lushnikov, Journal of Fluid Mechanics, v. 800, 557-594 (2016)

Approach 2: Analytical coupling of expansions near singularities in all sheets [1]

Equation for Stokes wave: $$y = -\frac{i}{2}(\tilde{z} - \bar{\tilde{z}}) = \frac{c^2}{2}\left(1 - \frac{1}{|z_u|^2}\right)$$

$$\zeta = \kappa + i\chi = \tan(\tfrac{w}{2})$$

$\Rightarrow$ Expansions in ***l*** th sheet

$$z_{l,+}(\zeta) = \sum_{j=0}^{\infty} \mathrm{i} e^{-\mathrm{i} j\pi/4} a_{+,l,j} (\zeta - \mathrm{i}\chi_c)^{j/2}, \quad l = 1, 2, \ldots,$$ - upper half-plane

$$z_{l,-}(\zeta) = \sum_{j=0}^{\infty} \mathrm{i} e^{-\mathrm{i} j\pi/4} a_{-,l,j} (\zeta + \mathrm{i}\chi_c)^{j/2}, \quad l = 1, 2, \ldots,$$ - lower half-plane

are coupled as follows:

[1] P. M. Lushnikov, Journal of Fluid Mechanics, **800**, 557-594 (2016)

Coupling of singularities at $\zeta = \pm i\chi_c$:

$$z_{l,+}(\zeta) = \sum_{j=0}^{\infty} \mathrm{i} e^{-\mathrm{i}j\pi/4} a_{+,l,j} (\zeta - \mathrm{i}\chi_c)^{j/2}, \quad l = 1, 2, \ldots,$$

$$z_{l,-}(\zeta) = \sum_{j=0}^{\infty} \mathrm{i} e^{-\mathrm{i}j\pi/4} a_{-,l,j} (\zeta + \mathrm{i}\chi_c)^{j/2}, \quad l = 1, 2, \ldots,$$

$$a_{+,2n+1,1} = -\frac{16c^2}{3\left(1-{\chi_c}^2\right)^2 a_{-,2n,3}(c^2 - a_{-,2n,0} + a_{+,2n+1,0})},$$

$$a_{+,2n+1,2} = \frac{2}{1-{\chi_c}^2} + \frac{128c^4}{9(1-{\chi_c}^2)^4 a_{-,2n,3}^2 (c^2 - a_{-,2n,0} - a_{+,2n+1,0})^3}$$
$$+ \frac{32c^2[-2\chi_c + (1-\chi_c^2)^2 a_{-,2n,4}]}{9(1-{\chi_c}^2)^4 a_{-,2n,3}^2 (c^2 - a_{-,2n,0} - a_{+,2n+1,0})},$$

$$a_{+,2n+1,3} = \ldots,$$

$$\ldots$$

$$a_{-,2n,1} = 0,$$

$$a_{-,2n,2} = \frac{-2}{1-{\chi_c}^2},$$

$$a_{-,2n,3} = \frac{16c^2}{3\left(1-\chi_c^2\right)^2 a_{+,2n-1,1}(c^2 - a_{+,2n-1,0} - a_{-,2n,0})},$$

$$a_{-,2n,4} = \frac{2\chi_c}{(1-{\chi_c}^2)^2} + \frac{4c^2}{(1-{\chi_c}^2)^2 (c^2 - a_{+,2n-1,0} - a_{-,2n,0})^2}$$
$$- \frac{8c^2[2 + (-1+\chi_c^2) a_{+,2n-1,2}]}{(1-{\chi_c}^2)^3 a_{+,2n-1,1}^2 (c^2 - a_{+,2n-1,0} - a_{-,2n,0})},$$

...

Is other type of singularity possible?

1. Assume coupling of singularities as power law:

$$z(w) = \sum_{n,m} c_{n,m}(w - w_1)^{n/2+m\alpha}$$

and

$$\bar{z}(w) = \sum_{n=0}^{\infty} d_n(w - w_1)^{n/2}$$

$\Rightarrow$ α is half-integer, i.e. no new solutions

2. If $\bar{z}(w)$ is analytic :

$$\bar{z}(w) = \sum_{n=0}^{\infty} d_n(w - w_1)^{n}$$

$\Rightarrow$ α only movable singularity is possible for $z(w)$ with α half-integer again.

3. Fixed singularity is possible but unlikely for

$$\bar{z}(w) = \sum_{n=0}^{\infty} d_n(w - w_1)^{n/2}$$

Conjecture how to obtain 2/3 power law of limiting Stokes wave from ½ power law singularities in the limit [1]

$$\chi_c \to 0$$

$$z \propto \left[(\zeta - \mathrm{i}\chi_c)^{1/2} + (-2\mathrm{i}\chi_c)^{1/2}\right] \sqrt{\alpha_1 \chi_c^{1/4} + \sqrt{(\zeta - \mathrm{i}\chi_c)^{1/2} + (-2\mathrm{i}\chi_c)^{1/2}}}$$

$$\times \sqrt{\alpha_3 \chi_c^{1/16} + \sqrt{\alpha_2 \chi_c^{1/8} + \sqrt{\alpha_1 \chi_c^{1/4} + \sqrt{(\zeta - \mathrm{i}\chi_c)^{1/2} + (-2\mathrm{i}\chi_c)^{1/2}}}}} \times \cdots$$

$$\alpha_1 = 0.0955383\ldots + i1.8351\ldots$$

$$\zeta \gg \chi_c \quad \Rightarrow \quad \boxed{z \propto \zeta^{1/2+1/8+1/32+\ldots} = \zeta^{2/3}}$$

[1] P. M. Lushnikov, Journal of Fluid Mechanics, **800**, 557-594 (2016)

Expression under the most inner square root:

$$g(\zeta) \equiv (\zeta - \mathrm{i}\chi_c)^{1/2} + (-2\mathrm{i}\chi_c)^{1/2}$$

Two branches at $\boldsymbol{\zeta = -\mathrm{i}\chi_c}$:

$$g_+(\zeta) = 2(-2\mathrm{i}\chi_c)^{1/2} + \frac{\zeta + \mathrm{i}\chi_c}{2(-2\mathrm{i}\chi_c)^{1/2}} + O(\zeta + \mathrm{i}\chi_c)^2$$

- no singularity of $\sqrt{g(\zeta)}$

$$g_-(\zeta) = -\frac{\zeta + \mathrm{i}\chi_c}{2(-2\mathrm{i}\chi_c)^{1/2}} + O(\zeta + \mathrm{i}\chi_c)^2$$

- singularity of $\sqrt{g(\zeta)}$ at $\boldsymbol{\zeta = -\mathrm{i}\chi_c}$

More details on solution

$$
\begin{aligned}
z &= \mathrm{i}\frac{c^2}{2} + c_1\chi_c^{1/6}\sqrt{\zeta - \mathrm{i}\chi_c} \\
&+ \frac{(3c)^{2/3}}{2} e^{-\mathrm{i}\pi/6}\left[(\zeta-\mathrm{i}\chi_c)^{1/2} + (-2\mathrm{i}\chi_c)^{1/2}\right]\sqrt{\alpha_1\chi_c^{1/4} + \sqrt{(\zeta-\mathrm{i}\chi_c)^{1/2} + (-2\mathrm{i}\chi_c)^{1/2}}} \\
&\times \sqrt{\alpha_3\chi_c^{1/16} + \sqrt{\alpha_2\chi_c^{1/8} + \sqrt{\alpha_1\chi_c^{1/4} + \sqrt{(\zeta-\mathrm{i}\chi_c)^{1/2} + (-2\mathrm{i}\chi_c)^{1/2}}}}} \\
&\times \sqrt{\alpha_{2n+1}\chi_c^{1/2^{2n+2}} + \sqrt{\alpha_{2n}\chi_c^{1/2^{2n+1}} + \sqrt{\ldots + \sqrt{\alpha_1\chi_c^{1/4} + \sqrt{(\zeta-\mathrm{i}\chi_c)^{1/2} + (-2\mathrm{i}\chi_c)^{1/2}}}}}} \\
&\times \cdots + \frac{(3c)^{2/3}}{2} e^{-\mathrm{i}\pi/6}\left[(\zeta-\mathrm{i}\chi_c)^{1/2} + (-2\mathrm{i}\chi_c)^{1/2}\right]\sqrt{\tilde{\alpha}_1\chi_c^{1/4} + \sqrt{(\zeta-\mathrm{i}\chi_c)^{1/2} + (-2\mathrm{i}\chi_c)^{1/2}}} \\
&\times \sqrt{\tilde{\alpha}_3\chi_c^{1/16} + \sqrt{\tilde{\alpha}_2\chi_c^{1/8} + \sqrt{\tilde{\alpha}_1\chi_c^{1/4} + \sqrt{(\zeta-\mathrm{i}\chi_c)^{1/2} + (-2\mathrm{i}\chi_c)^{1/2}}}}} \\
&\times \sqrt{\tilde{\alpha}_{2n+1}\chi_c^{1/2^{2n+2}} + \sqrt{\tilde{\alpha}_{2n}\chi_c^{1/2^{2n+1}} + \sqrt{\ldots + \sqrt{\tilde{\alpha}_1\chi_c^{1/4} + \sqrt{(\zeta-\mathrm{i}\chi_c)^{1/2} + (-2\mathrm{i}\chi_c)^{1/2}}}}}}
\end{aligned}
$$

× ···+h.o.t.

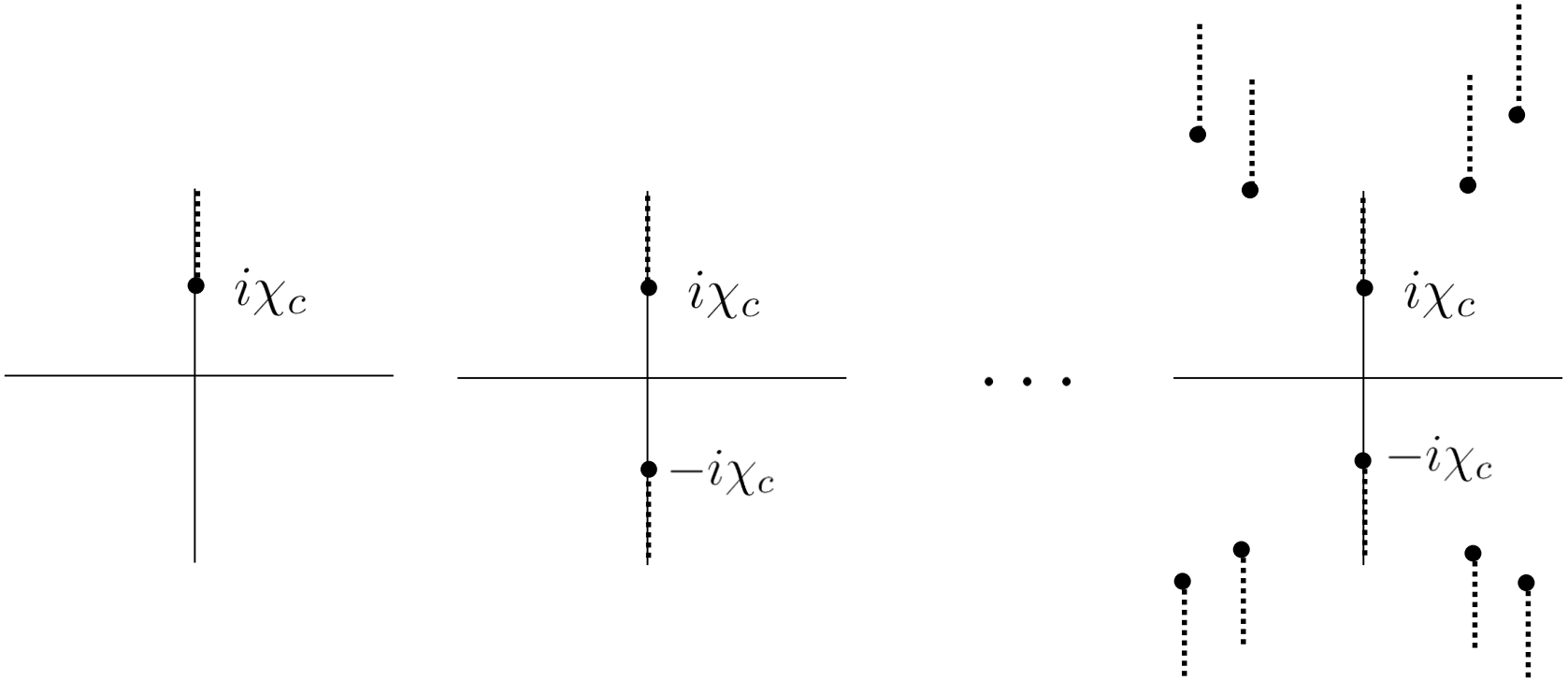
$\alpha_1 \simeq -0.0955383 - \mathrm{i}\,1.8351$ - determined by position of first off-axis singularity

Location of singularities in infinite numbers of sheets of Riemann surface[1]

First (physical) sheet

Second (non-physical) sheet

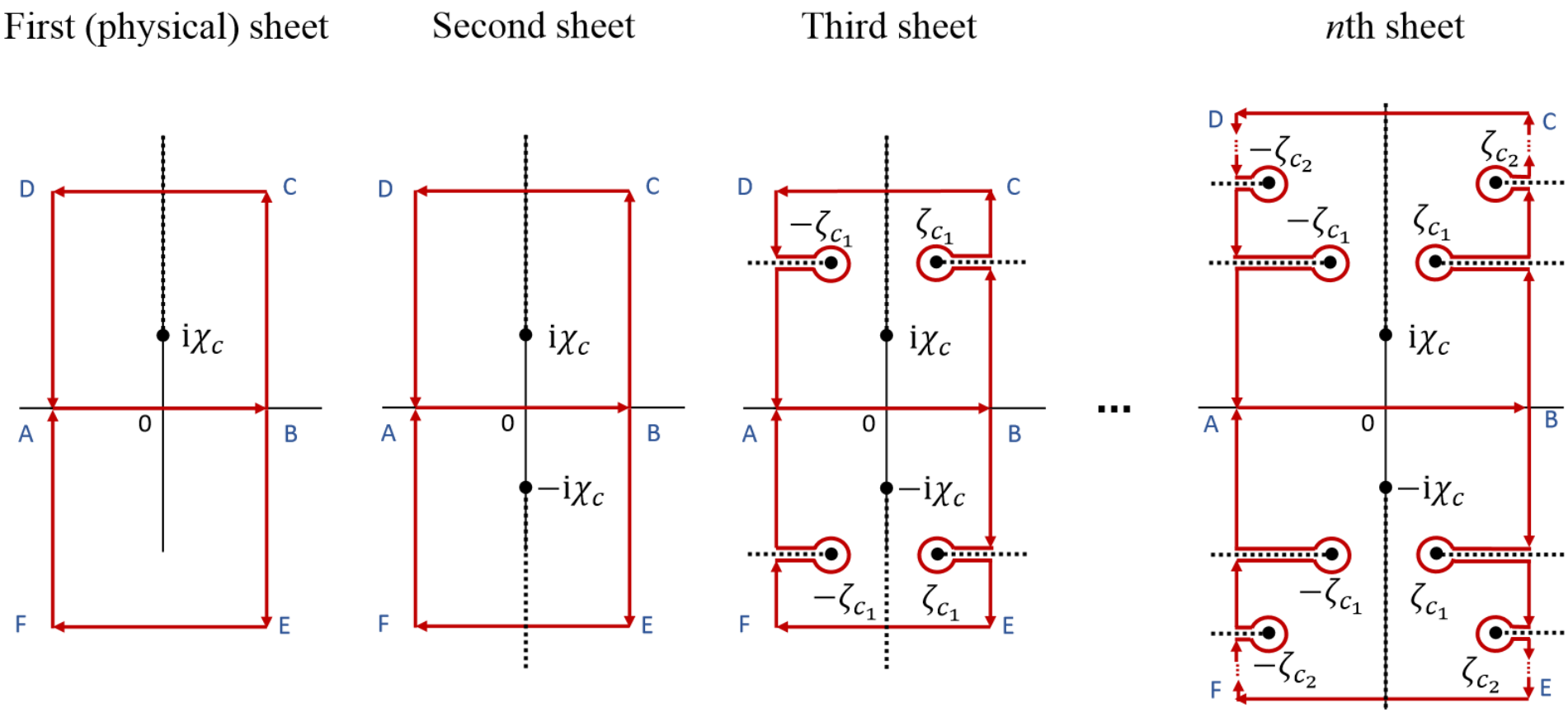
Third and higher sheets



All singularities are square roots[1]

[1]P. M. Lushnikov, Journal of Fluid Mechanics, **800**, 557-594 (2016)

$\alpha_1 \simeq -0.0955383 - \mathrm{i}\,1.8351$ - and all others constants $\alpha_2,\ \alpha_3,\ \alpha_4, \ldots$ are determined by positions of off-axis singularities



Ratio of coefficients of analytical and numerical series:

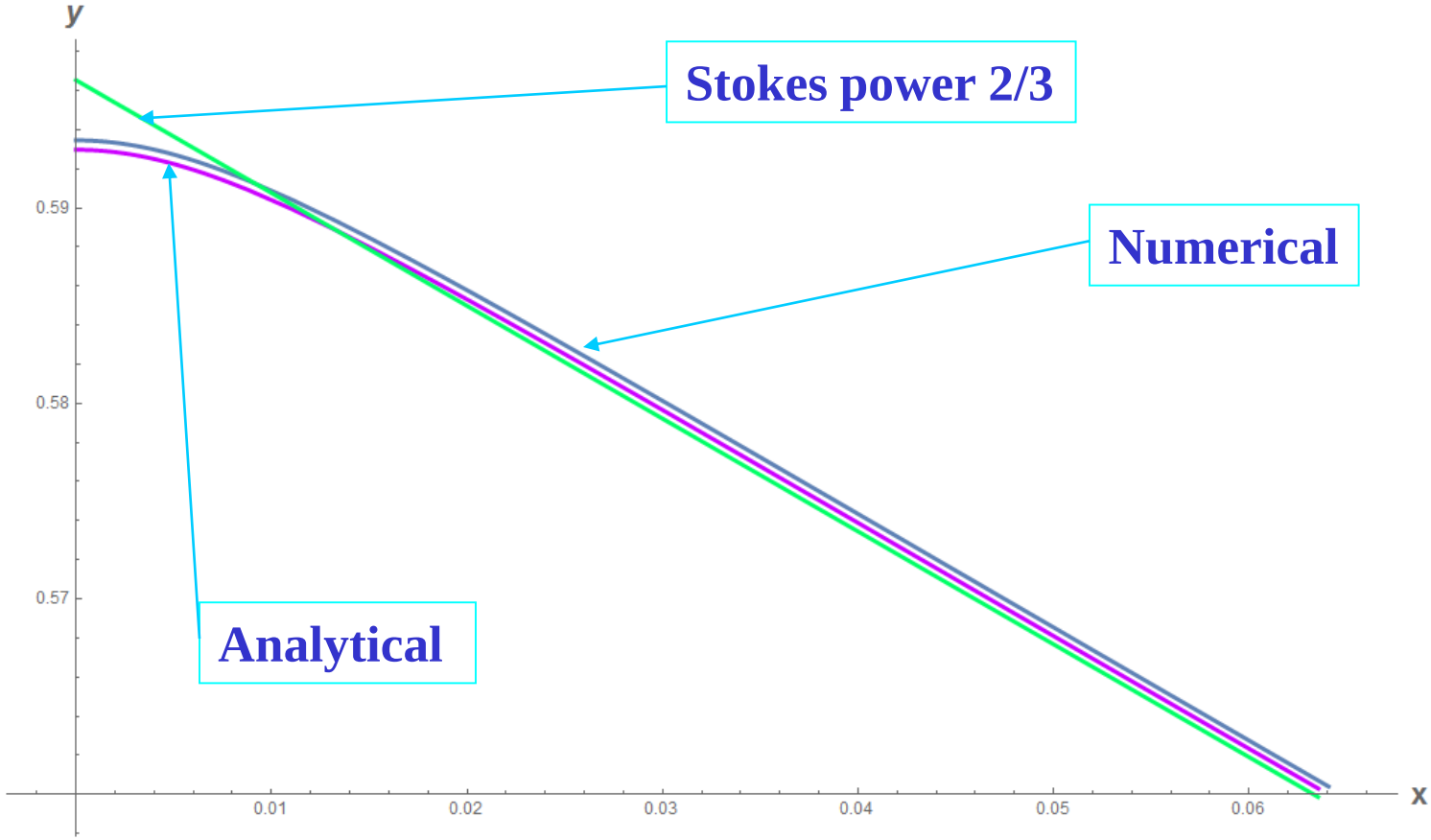
(a) At $\zeta = \mathrm{i}\,\chi_c$ (both integer and half-integer powers):

$$(1, 1, 1, 1, 1.00898 + \mathrm{i}\,0.0633632, 1.02396 + \mathrm{i}\,0.0948505, 1.03738 + \mathrm{i}\,0.112144,$$
$$1.04785 + \mathrm{i}\,0.12241, 1.05576 + \mathrm{i}\,0.128944, 1.06175 + \mathrm{i}\,0.133369, 1.06636 + \mathrm{i}\,0.136528,$$
$$1.06998 + \mathrm{i}\,0.138884, \ldots)$$

(b) At $\zeta = -\mathrm{i}\,\chi_c$ (only integer powers):

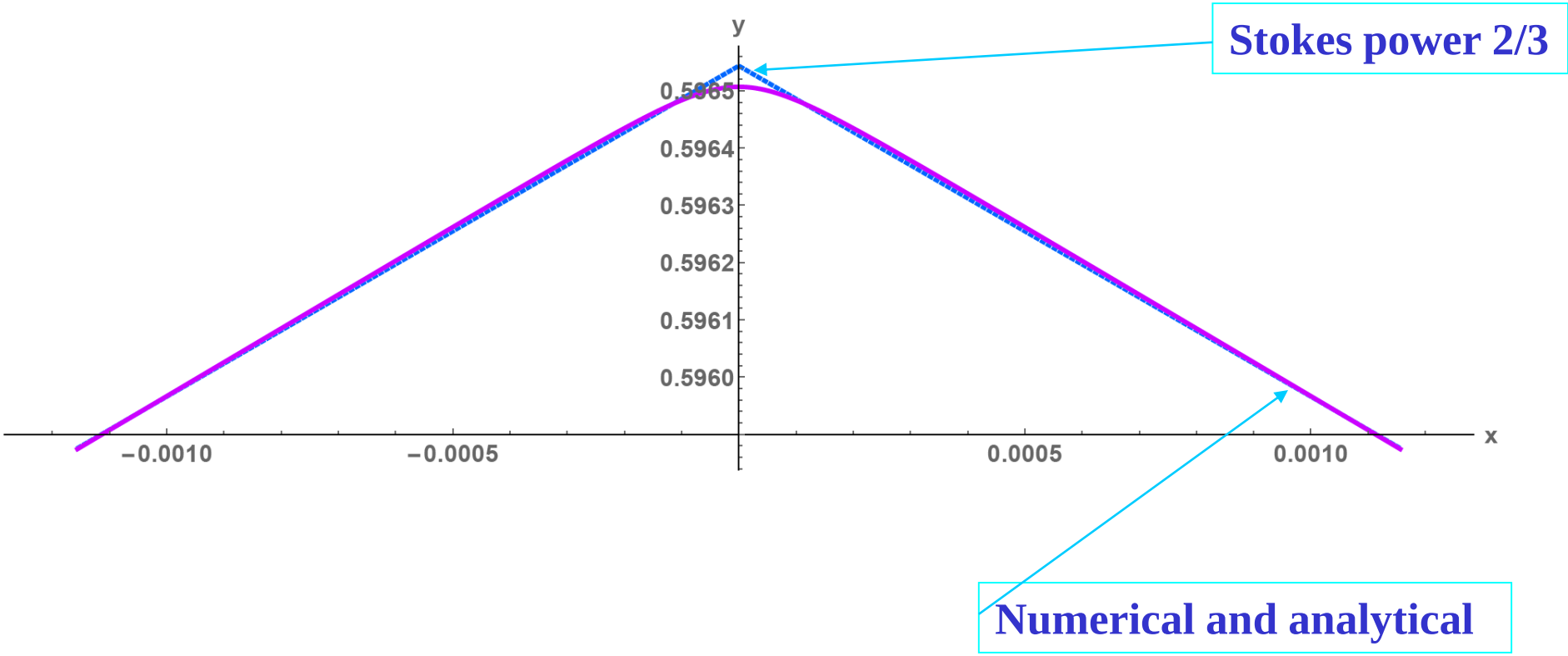
$$(0.999996, 1.00012 + \mathrm{i}\,0.00167653, 0.999884 - \mathrm{i}\,0.00307163, 0.999892 - \mathrm{i}\,0.000839551,$$
$$0.999937 - \mathrm{i}\,0.000373147, 0.99996 - \mathrm{i}\,0.00020832, 0.999973 - \mathrm{i}\,0.000132391,$$
$$0.999981 - \mathrm{i}\,0.0000914145, 0.999985 - \mathrm{i}\,0.0000668552, 0.999989 - \mathrm{i}\,0.0000509973,$$
$$0.999991 - 0.0000401725, 0.999993 - 0.000032458, 0.999994 - 0.0000267681, \ldots)$$

Comparison of analytical, numerical and Stokes power 2/3 solutions



$\chi_c \sim 10^{-3}$

For smaller χ_c analytical and numerical are visually indistinguishable”



$\chi_c \sim 10^{-6}$

Different approaches for numerics

1. Fourier transform on uniform grid requires

Asymptotic $k \to -\infty$ of the Fourier series $\tilde{z}(u) = \sum_{k=0}^{-\infty} \hat{\tilde{z}}_k \exp(\mathrm{i}ku)$ of

$\tilde{z} \simeq c_1 (w - \mathrm{i}v_c)^\beta$ is given by $|\hat{\tilde{z}}_k| \propto |k|^{-1-\beta} e^{-|k|v_c}$ $\boxed{\beta = 1/2}$

2. Scaling of the error of Pade approximation $err_\infty := \max_{-\infty<\zeta<\infty} |\tilde{z}(\zeta) - \tilde{z}(\zeta)_{pade}|$

$$err_\infty \propto e^{-p(v_c)N}$$

$$p(v_c) \propto v_c^{1/6}$$

$\Rightarrow$

Pade approximation is many order more efficient for small $\boldsymbol{v_c}$

Conformal transformation method:

For general time-dependent problem we use Fourier transform which has uniform grid in the new auxiliary variable q which corresponds to highly non-uniform grid in u.

Additional conformal transformation between u and q:

$$q = 2\arctan\left[\frac{1}{L}\tan\frac{u}{2}\right]$$

Parameter: $L \ll 1$

$$\Rightarrow \qquad u = 2\arctan\left[L\tan\frac{q}{2}\right]$$

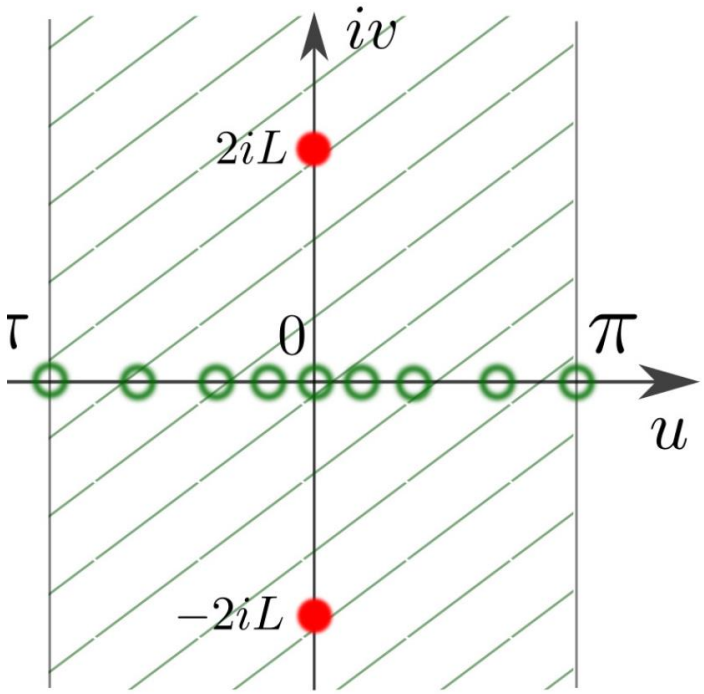
Uniform grid in q $\Rightarrow$ nonuniform grid in u

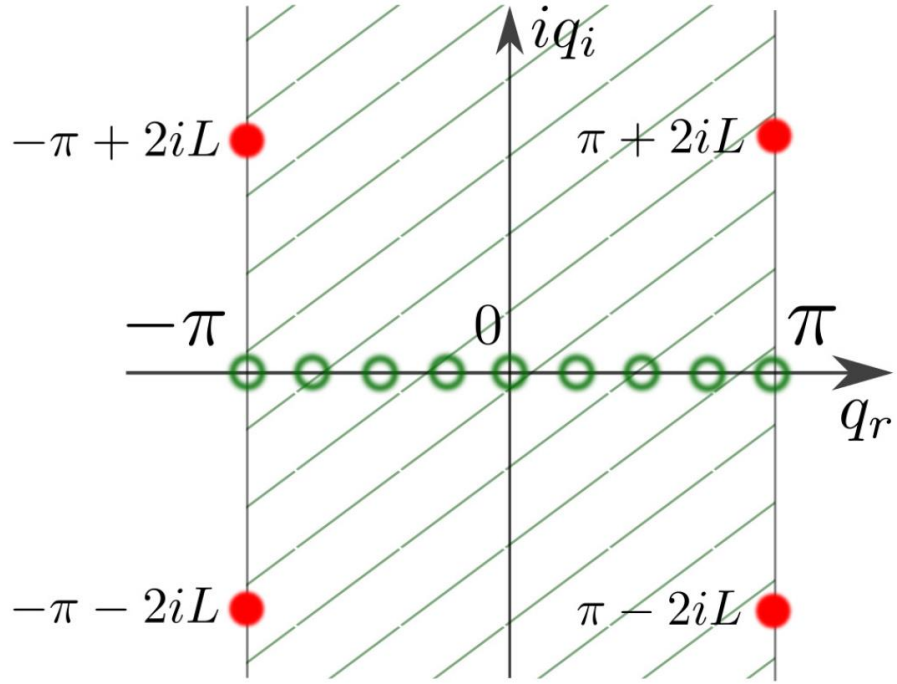
$$\Delta q = q_u \Delta u = \frac{1 + L^2 + (1 - L^2)\cos q}{2L}\Delta u$$

For $u \to 0$: $\Delta u \simeq L\Delta q$

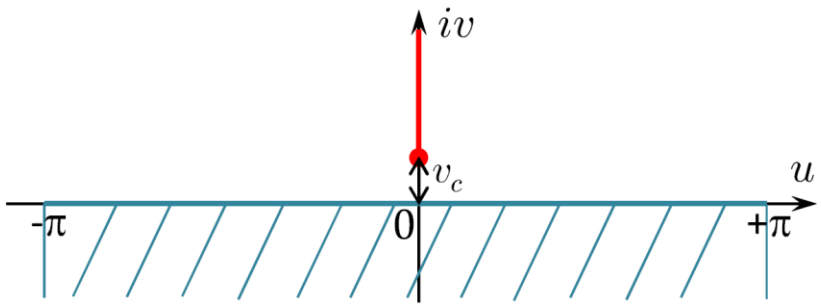
Singularities of conformal map

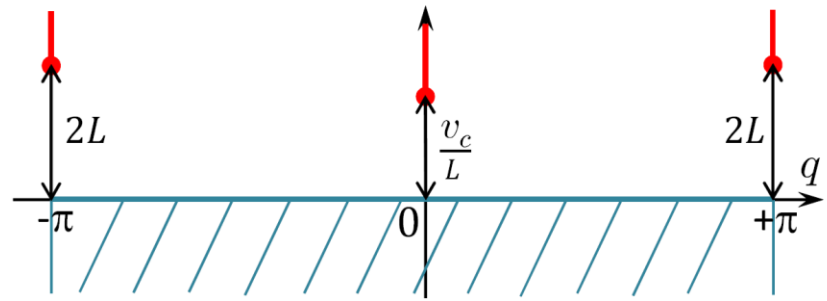
$$q = 2\arctan\left[\frac{1}{L}\tan\frac{u}{2}\right]$$





For branch point of water wave at $w = iv_c$ $\Rightarrow$ Branch point at $q = iv_c/L$





Location of branch cuts of the transformation:

$$w = 2\arctan\frac{\mathrm{i}}{L} = \pm\pi + 2\mathrm{i}L + O(L^3)$$

Transformation moves the singularity upwards: $v_c \to v_c/L$

The optimal choice for the fastest spectral convergence is when $2L = v_c/L$

$$\Rightarrow \qquad L_{optimal} \simeq \left(\frac{v_c}{2}\right)^{1/2}$$

which ensures that branch point is pushed up to $\boxed{\mathrm{i}(2v_c)^{1/2}}$

But how to work with the projectors of dynamics equation

Projectors through integrals in variable u:

$$\hat{P}_u^+ f = \frac{1}{2\pi \mathrm{i}} \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} \frac{f(u')du'}{u' - u - \mathrm{i}0 + 2\pi n} = \frac{1}{4\pi \mathrm{i}} \int_{-\pi}^{\pi} \frac{f(u')du'}{\tan\frac{u'-u-\mathrm{i}0}{2}}$$

$$\hat{P}_u^- f = -\frac{1}{2\pi \mathrm{i}} \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} \frac{f(u')du'}{u' - u + \mathrm{i}0 + 2\pi n} = -\frac{1}{4\pi \mathrm{i}} \int_{-\pi}^{\pi} \frac{f(u')du'}{\tan\frac{u'-u+\mathrm{i}0}{2}}$$

Change of variables from u to q :

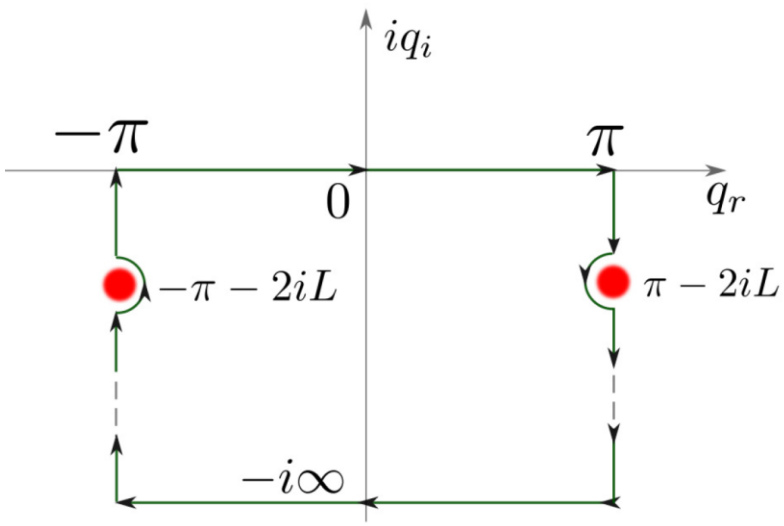
$$\hat{P}_u^+ f = \frac{1}{4\pi \mathrm{i}} \int_{-\pi}^{\pi} \frac{f(q')dq'}{\tan\frac{u'-u-\mathrm{i}0}{2}} u_{q'} = \frac{1}{4\pi \mathrm{i}} \int_{-\pi}^{\pi} \frac{f(q')\left[1 + \tan\frac{u'-\mathrm{i}0}{2}\tan\frac{u}{2}\right]dq'}{\tan\frac{u'-\mathrm{i}0}{2} - \tan\frac{u}{2}} \frac{L}{\cos^2\frac{q'}{2}\left(1 + L^2\tan^2\frac{q'}{2}\right)}$$

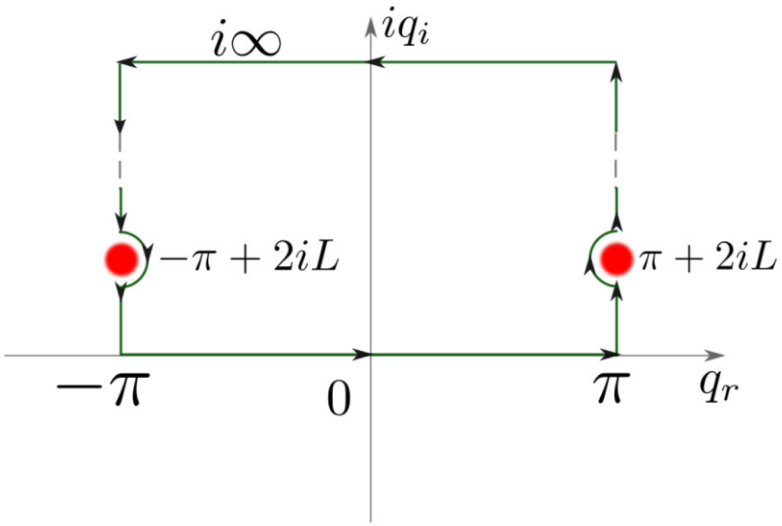
$$= \frac{1}{4\pi \mathrm{i}} \int_{-\pi}^{\pi} \frac{f(q')\left[1 + L^2\tan\frac{q'-\mathrm{i}0}{2}\tan\frac{q}{2}\right]dq'}{\tan\frac{q'-\mathrm{i}0}{2} - \tan\frac{q}{2}} \frac{1}{\cos^2\frac{q'}{2}\left(1 + L^2\tan^2\frac{q'}{2}\right)}$$

and similar for $\hat{P}_u^- f$

Zeros of denominator are at $q' = q + \mathrm{i}0$ and $q' = \pm 2\arctan\frac{\mathrm{i}}{L}$

Integration contours





We split *f* into parts analytic in upper and lower half-planes of *q*:

$$f(u) \equiv f(q) = f^{+,q}(q) + f^{-,q}(q) + f_{0,q}$$

and calculate integrals from previous slide in either upper or lower complex half-planes to ensure the convergence of each term which gives

$$\hat{P}_u^+ f(q) = \boxed{\frac{f_{0,q}}{2} + f^{+,q}(q)} - \frac{1}{2} f^{+,q}\left(2\arctan\frac{\mathrm{i}}{L}\right) + \frac{1}{2} f^{-,q}\left(-2\arctan\frac{\mathrm{i}}{L}\right)$$

$$\hat{P}_u^- f(q) = \boxed{\frac{f_{0,q}}{2} + f^{-}(q)} - \frac{1}{2} f^{-,q}\left(-2\arctan\frac{\mathrm{i}}{L}\right) + \frac{1}{2} f^{+,q}\left(2\arctan\frac{\mathrm{i}}{L}\right).$$

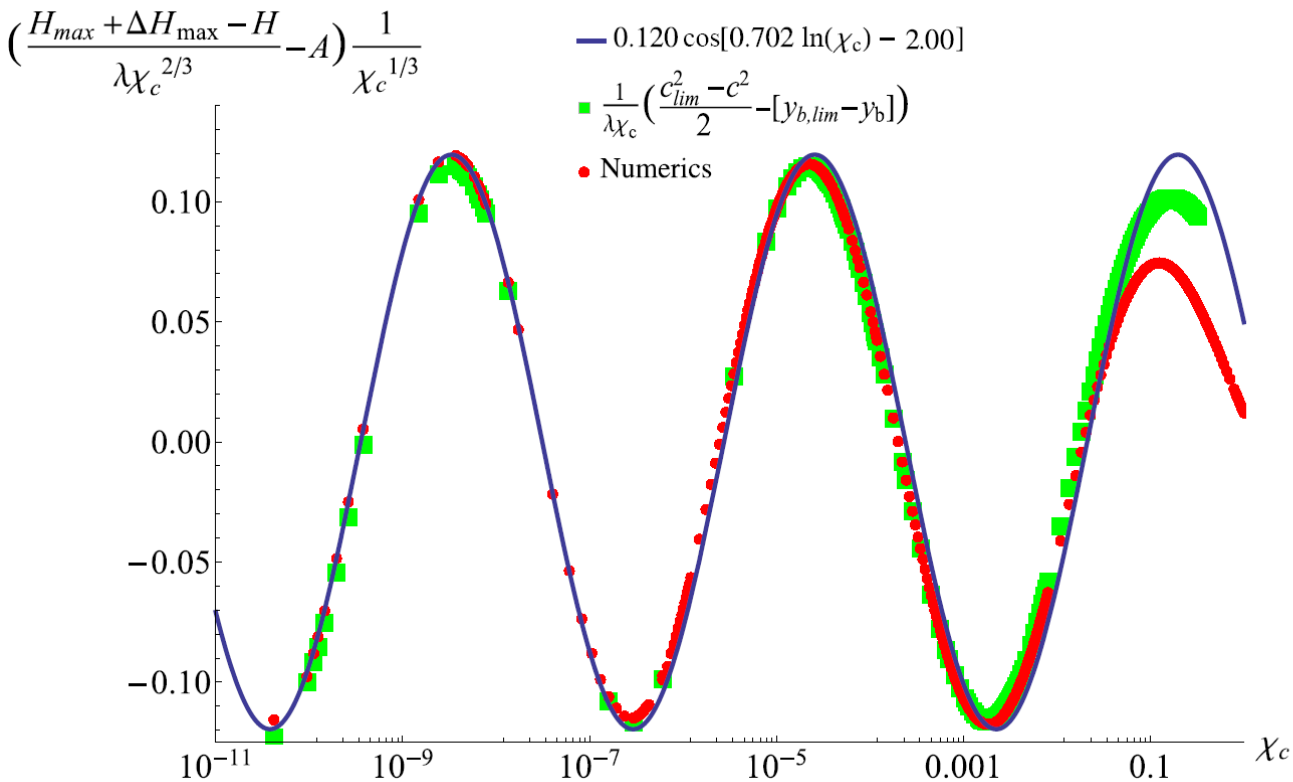
Projector operators in q variable

For real-valued function *f(q)*:

$$f^{+,q}\left(2\arctan\frac{\mathrm{i}}{L}\right) = \overline{f^{-,q}\left(-2\arctan\frac{\mathrm{i}}{L}\right)}$$

- calculated by analytical continuation of Fourier series

Results of conformal method for Stokes wave [1] **:** Comparison with matched asymptotics of Ref.[2]



---

[1] P. M. Lushnikov, S.A. Dyachenko and D.A. Silantyev, Submitted to Proc. Roy. Soc. A (2017)

[2] M. S. Longuet-Higgins and M. J. H. Fox, Theory of the almost-highest wave. Part 2. Matching and analytic extension, J. Fluid Mech. **85(4)**, 769 786 (1978).

Velocity Oscillations

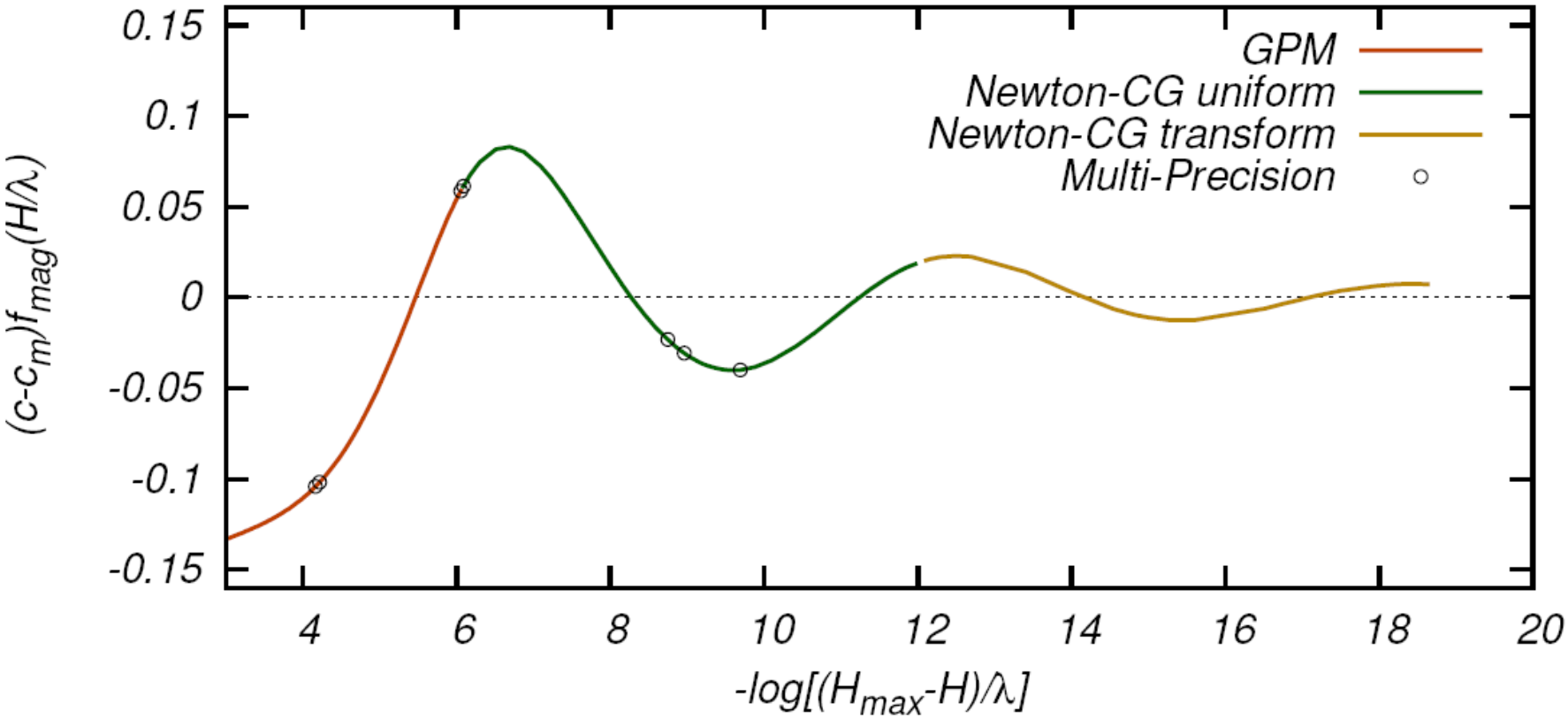


Figure : Oscillations of dimensionless velocity c: the figure is scaled by magnification function $f_{mag}\left(\frac{H}{\lambda}\right) = \frac{1}{(30\frac{H_{max}-H}{\lambda})^{1.3}+1}$ to show all simulation data in a single graph while stressing obtained oscillations.[4]

Generalization of conformal map to resolve multiple singularities [1]

$$q(u) = \sum_{j=1}^{N} 2\beta_j \arctan\left[\frac{1}{L_j}\left(\tan\frac{u}{2} - \tan\frac{u_j}{2}\right)\right] \qquad \sum_{j=1}^{N} \beta_j = 1$$

$$L_j > 0,\ \beta_j > 0 \qquad -\pi < u_j < \pi$$

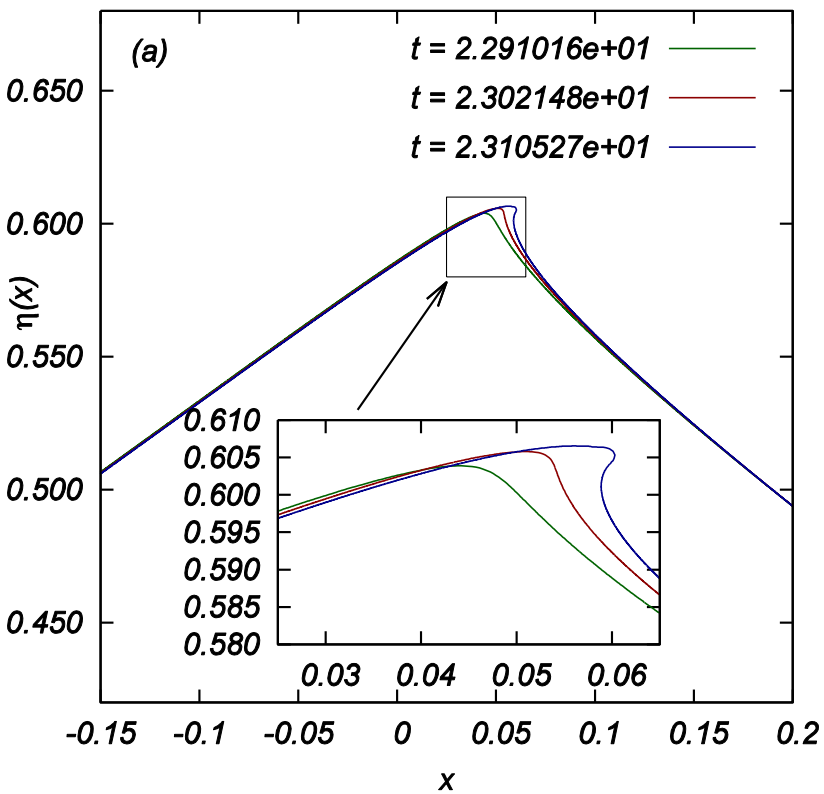
Jacobian is positive-definite:

$$q_u = \sum_{j=1}^{N} \frac{2\beta_j}{L_j \cos^2\frac{u}{2}\left(1 + \frac{\left[\tan\frac{u}{2} - \tan\frac{u_j}{2}\right]^2}{L_j^2}\right)}$$

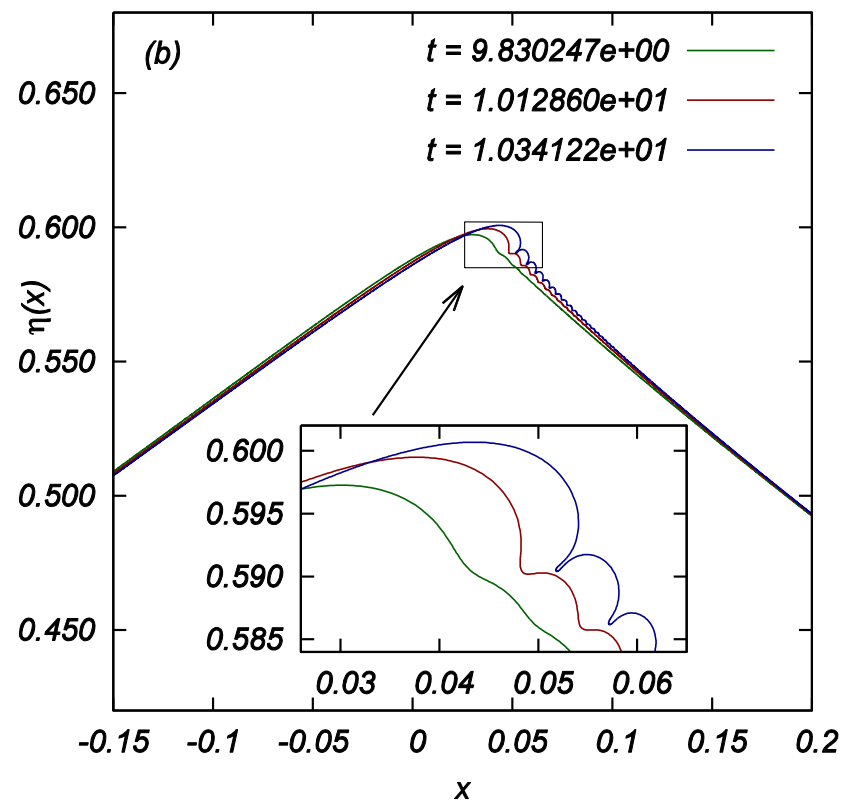
[1] P. M. Lushnikov, S.A. Dyachenko and D.A. Silantyev, Submitted to Proc. Roy. Soc. A (2017)

Conclusion: Practical calculations demonstrated speed up of simulations up to 10^6 times

1. Stokes wave simulations with semi-analytic Pade quadratures
2. Time-dependent simulations



$\sigma = 0$



$\sigma = 1.6 \cdot 10^{-5}$

Conclusion and future directions

- Analytical properties of Stokes wave in the first sheet of Riemann surface are fully determined by a single branch cut and the solution for Stokes waves in the first sheet is reduced to the evaluation of integral along that branch cut

$$\tilde{z}(\zeta) = z_0 + \int_{\chi_c}^{1} \frac{\rho(\chi')d\chi'}{\zeta - i\chi'} \qquad \zeta = \kappa + i\chi = \tan(\tfrac{w}{2})$$

- Conjecture that locations of all branch points are determined by the infinite number of embedded square roots which recovers Stokes limiting wave solution with 2/3 singularity

- Pade-type quadrature is constructed using analytical information about the jump at branch to solve the closed equations for Stokes wave either avoiding Fourier transform or using non-uniform grid.

-Ultimate goal for the future is the description of 2D hydrodynamics with free surface through the dynamics of branch cuts

References

[1]S.A. Dyachenko, P.M. Lushnikov, and A.O. Korotkevich, JETP Letters, **98**, 675-679 (2014).

[2]S.A. Dyachenko, P.M. Lushnikov, and A.O. Korotkevich, Stud. Appl. Math., **137**, 419-472 (2016)

[3]P. M. Lushnikov, Journal of Fluid Mechanics, **800**, 557-594 (2016)

[4]P. M. Lushnikov, S.A. Dyachenko and D.A. Silantyev, Submitted to Proc. Roy. Soc. A (2017)